General notes - 1A NST Maths-A
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Francisco Berkemeier

Notes 1-5 are relevant to the topics and exercises covered in the 2023 Michaelmas term.
Note 6 is relevant to the topics and exercises covered in the 2024 Lent term.

Contents

1 On reciprocal bases

1.1 Definition

Consider a vector space V with a basis $B = \{\boldsymbol{e}_i\}_{i\in I}$, where I is a set of indexes. The *reciprocal* (or *dual*) basis of V is the set $B^* = \{\boldsymbol{e}^i\}_{i\in I}$ with the same index set I such that B and B^* form a *biorthogonal system*, that is,

$$\boldsymbol{e}_i \cdot \boldsymbol{e}^j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Alternatively, one may write $\boldsymbol{e}_i \cdot \boldsymbol{e}^j = \delta_{ij}$, where δ_{ij} is the *Kronecker delta.*

1.2 Examples

Example 1.1 (Basis components) Take $V = \mathbb{R}^3$ and consider the basis $B = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$. Take $B^* = \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}$ as the reciprocal basis of B. Consider any vector $\boldsymbol{d} \in V$ written as $\boldsymbol{d} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c}$. Applying the scalar product with $\boldsymbol{A}$ in both sides of this equation leads to

$$\boldsymbol{d} \cdot \boldsymbol{A} = \alpha(\boldsymbol{a} \cdot \boldsymbol{A}) + \beta(\boldsymbol{b} \cdot \boldsymbol{A}) + \gamma(\boldsymbol{c} \cdot \boldsymbol{A}) \;\Leftrightarrow\; \boldsymbol{d} \cdot \boldsymbol{A} = \alpha$$

since, by definition, $\boldsymbol{a} \cdot \boldsymbol{A} = 1$, and $\boldsymbol{b} \cdot \boldsymbol{A} = \boldsymbol{c} \cdot \boldsymbol{A} = 0$. Similarly, $\boldsymbol{d} \cdot \boldsymbol{B} = \beta$ and $\boldsymbol{d} \cdot \boldsymbol{C} = \gamma$. Hence, the coefficients α, β, and γ for any vector $\boldsymbol{d}$ written in terms of a basis $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ can be obtained via the scalar product with the vector of the corresponding reciprocal basis.

Example 1.2 (Relation with scalar triple product) The scalar triple product can hint at potential candidates for the reciprocal basis vectors $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ in the previous example. The vector product here is written as $\times$ (unlike the $\wedge$ notation from the notes). Recall the property of the scalar triple product

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = -\boldsymbol{a} \cdot (\boldsymbol{c} \times \boldsymbol{b}) = -\boldsymbol{b} \cdot (\boldsymbol{d} \times \boldsymbol{c}) = -\boldsymbol{c} \cdot (\boldsymbol{b} \times \boldsymbol{a}).$$

Consider $\boldsymbol{A}$ first. We need $\boldsymbol{a} \cdot \boldsymbol{A} = 1$, $\boldsymbol{b} \cdot \boldsymbol{A} = 0$ and $\boldsymbol{c} \cdot \boldsymbol{A} = 0$. A natural choice could be

$$\boldsymbol{A} = \frac{\boldsymbol{b} \times \boldsymbol{c}}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})}$$

since, using the property of the scalar triple product stated before,

$$\begin{aligned}
\boldsymbol{a} \cdot \boldsymbol{A} &= \frac{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})} = 1 \\
\boldsymbol{b} \cdot \boldsymbol{A} &= \frac{\boldsymbol{b} \cdot (\boldsymbol{b} \times \boldsymbol{c})}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})} = \frac{\boldsymbol{c} \cdot (\boldsymbol{b} \times \boldsymbol{b})}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})} = 0 \\
\boldsymbol{c} \cdot \boldsymbol{A} &= \frac{\boldsymbol{c} \cdot (\boldsymbol{b} \times \boldsymbol{c})}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})} = \frac{\boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{c})}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})} = 0.
\end{aligned}$$

Hence, with

$$\begin{aligned}
\boldsymbol{B} &= \frac{\boldsymbol{c} \times \boldsymbol{a}}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})} \\
\boldsymbol{C} &= \frac{\boldsymbol{a} \times \boldsymbol{b}}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})}
\end{aligned}$$

it follows that

$$\begin{aligned}
\boldsymbol{b} \cdot \boldsymbol{B} &= \frac{\boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a})}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})} = \frac{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})} = 1 \\
\boldsymbol{c} \cdot \boldsymbol{C} &= \frac{\boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b})}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})} = \frac{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})} = 1.
\end{aligned}$$

A similar argument shows that $\boldsymbol{b} \cdot \boldsymbol{A} = \boldsymbol{b} \cdot \boldsymbol{C} = \boldsymbol{c} \cdot \boldsymbol{A} = \boldsymbol{c} \cdot \boldsymbol{B} = 0$.

Example 1.3 (Problem 8(b) from Section C) In this problem, we have

$$\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{c} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \boldsymbol{d} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and we are asked to write the vector $\boldsymbol{d}$ in terms of the basis $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ by using the scalar triple product. From Example 1.2, we have that, via the scalar triple product, the reciprocal basis is given by

$$\begin{aligned}
\boldsymbol{A} &= \frac{\boldsymbol{b} \times \boldsymbol{c}}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})} = \frac{1}{5}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \\
\boldsymbol{B} &= \frac{\boldsymbol{c} \times \boldsymbol{a}}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})} = \frac{1}{5}\begin{pmatrix} -3 \\ -1 \\ 5 \end{pmatrix} \\
\boldsymbol{C} &= \frac{\boldsymbol{a} \times \boldsymbol{b}}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})} = \frac{1}{5}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}
\end{aligned}$$

where $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = 5$. Hence, from Example 1.1,

$$\alpha = \boldsymbol{d} \cdot \boldsymbol{A} = \frac{3}{5}, \beta = \boldsymbol{d} \cdot \boldsymbol{B} = \frac{1}{5}, \gamma = \boldsymbol{d} \cdot \boldsymbol{C} = \frac{1}{5}$$

and thus $\boldsymbol{d} = \frac{3}{5}\boldsymbol{a} + \frac{1}{5}\boldsymbol{b} + \frac{1}{5}\boldsymbol{c}$.

2 Shortest distance of a line from a line

We are interested in describing the shortest distance d between two non-parallel lines L_1 and L_2 in $\mathbb{R}^3$. Assume these are given by

$$\boldsymbol{r} = \boldsymbol{a}_1 + \lambda\hat{\boldsymbol{t}_1}$$
$$\boldsymbol{r} = \boldsymbol{a}_2 + \mu\hat{\boldsymbol{t}_2}$$

respectively, for $\lambda, \mu \in \mathbb{R}$. We consider Figure 23 in Section 1.7.3 of the main notes, where the shortest distance d is between two points X and Y. We are essentially interested in describing the vector $\overrightarrow{XY}$ in two different ways, equating such expressions and solving for d.

First, it is straightforward to see that $\overrightarrow{XY} = d\hat{\boldsymbol{u}}$, where $\hat{\boldsymbol{u}}$ is a unit vector that should be perpendicular to both leading vectors $\hat{\boldsymbol{t}_1}$ and $\hat{\boldsymbol{t}_2}$, thus given by $\hat{\boldsymbol{u}} = (\hat{\boldsymbol{t}_1} \times \hat{\boldsymbol{t}_2})/|\hat{\boldsymbol{t}_1} \times \hat{\boldsymbol{t}_2}|$.

Second, we may write $\overrightarrow{XY}$ by looking at the triangle XYO and using the points on each line

$$\overrightarrow{XY} = \overrightarrow{OY} - \overrightarrow{OX} = (\boldsymbol{a}_2 + \mu\hat{\boldsymbol{t}_2}) - (\boldsymbol{a}_1 + \lambda\hat{\boldsymbol{t}_1}) = \boldsymbol{a}_2 - \boldsymbol{a}_1 + \mu\hat{\boldsymbol{t}_2} - \lambda\hat{\boldsymbol{t}_1}$$

for some specific λ and μ.

Finally, equating the expressions for $\overrightarrow{XY}$ and applying the scalar product with $\hat{\boldsymbol{t}_1} \times \hat{\boldsymbol{t}_2}$ on either side leads to

$$\begin{aligned}
& d\frac{\hat{\boldsymbol{t}_1} \times \hat{\boldsymbol{t}_2}}{|\hat{\boldsymbol{t}_1} \times \hat{\boldsymbol{t}_2}|} = \boldsymbol{a}_2 - \boldsymbol{a}_1 + \mu\hat{\boldsymbol{t}_2} - \lambda\hat{\boldsymbol{t}_1} \\
\Leftrightarrow\ & d\frac{(\hat{\boldsymbol{t}_1} \times \hat{\boldsymbol{t}_2}) \cdot (\hat{\boldsymbol{t}_1} \times \hat{\boldsymbol{t}_2})}{|\hat{\boldsymbol{t}_1} \times \hat{\boldsymbol{t}_2}|} = (\boldsymbol{a}_2 - \boldsymbol{a}_1) \cdot (\hat{\boldsymbol{t}_1} \times \hat{\boldsymbol{t}_2}) + (\mu\hat{\boldsymbol{t}_2} - \lambda\hat{\boldsymbol{t}_1}) \cdot (\hat{\boldsymbol{t}_1} \times \hat{\boldsymbol{t}_2}) \\
\Leftrightarrow\ & d\frac{|\hat{\boldsymbol{t}_1} \times \hat{\boldsymbol{t}_2}|^2}{|\hat{\boldsymbol{t}_1} \times \hat{\boldsymbol{t}_2}|} = (\boldsymbol{a}_2 - \boldsymbol{a}_1) \cdot (\hat{\boldsymbol{t}_1} \times \hat{\boldsymbol{t}_2}) \\
\Leftrightarrow\ & d = \frac{(\boldsymbol{a}_2 - \boldsymbol{a}_1) \cdot (\hat{\boldsymbol{t}_1} \times \hat{\boldsymbol{t}_2})}{|\hat{\boldsymbol{t}_1} \times \hat{\boldsymbol{t}_2}|}
\end{aligned}$$

where we have used the fact that $\hat{\boldsymbol{t}_1}$ and $\hat{\boldsymbol{t}_2}$ are parallel to $\lambda\hat{\boldsymbol{t}_1}$ and $\mu\hat{\boldsymbol{t}_2}$, respectively, and so

$$(\mu\hat{\boldsymbol{t}_2} - \lambda\hat{\boldsymbol{t}_1}) \cdot (\hat{\boldsymbol{t}_1} \times \hat{\boldsymbol{t}_2}) = \mu\hat{\boldsymbol{t}_2} \cdot (\hat{\boldsymbol{t}_1} \times \hat{\boldsymbol{t}_2}) - \lambda\hat{\boldsymbol{t}_1} \cdot (\hat{\boldsymbol{t}_1} \times \hat{\boldsymbol{t}_2}) = 0 - 0 = 0.$$

We may also think about this scalar product as the projection onto $\hat{\boldsymbol{u}} = (\hat{\boldsymbol{t}_1} \times \hat{\boldsymbol{t}_2})/|\hat{\boldsymbol{t}_1} \times \hat{\boldsymbol{t}_2}|$, which gives the same answer.

3 Integration

3.1 Absolute values

Consider the expression

$$\int \frac{-1}{x-1}\,dx = -\int \frac{1}{x-1}\,dx = -\log|x-1| + C. \tag{1}$$

On the other hand, we also have, we also have that

$$\int \frac{-1}{x-1}\,dx = \int \frac{1}{1-x}\,dx = -\int \frac{-1}{1-x}\,dx = -\log|1-x| + C. \tag{2}$$

Naturally, $|x-1| = |1-x|$.

3.2 Substitution

Example 3.1 (Problem 10(c) from Section H) We have

$$\int \frac{-2}{x^2-1}\,dx. \tag{3}$$

There are two ways of solving this exercise. First, we can factor $x^2 - 1 = (x+1)(x-1)$ to get, by partial fractions,

$$\int \frac{-2}{x^2-1}\,dx = \int \frac{1}{x+1}\,dx + \int \frac{-1}{x-1}\,dx = \log|x+1| - \log|1-x| + C = \log\left|\frac{1+x}{1-x}\right| + C \tag{4}$$

Alternatively, one could use the substitution $x = \tanh u$ to get

$$\int \frac{-2}{x^2-1}\,dx = 2\int \frac{\text{sech}^2 u}{1-\tanh^2 u}\,du = 2u + C = 2\tanh^{-1} x + C \tag{5}$$

which only holds for $|x| < 1$. In this interval, the identity $2\tanh^{-1} x = \log\left(\frac{1+x}{1-x}\right)$ holds. However, only the first answer is correct, because the substitution only holds for $|x| < 1$ and therefore we are restricting the evaluation of the integral to this specific domain. For all $x \in \mathbb{R}\backslash\{-1, 1\}$, an alternative way is to write (Figure 1)

$$\int \frac{-2}{x^2-1}\,dx = \log\left|\frac{1+x}{1-x}\right| + C = \begin{cases} 2\tanh^{-1} x + C, & \text{for } |x| < 1 \\ 2\coth^{-1} x + C, & \text{for } |x| > 1 \end{cases} \tag{6}$$

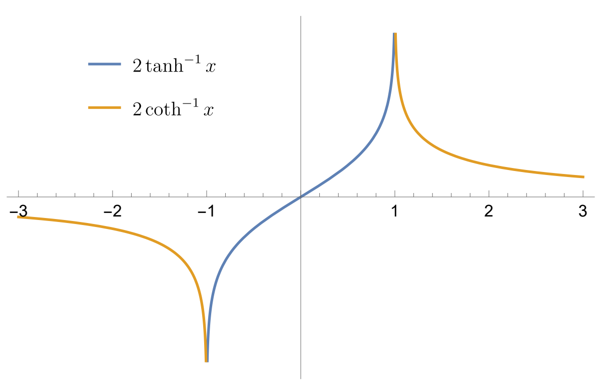

Figure 1: Plot of $\log\left|\frac{1+x}{1-x}\right|$.

Example 3.2 (Problem 11(b) from Section H) We are asked to find the indefinite integral

$$I = \int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx. \tag{7}$$

Using the substitution $u = e^x$, we get $du = e^x dx$ and thus

$$I = \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) + C = \sin^{-1}(e^x) + C. \tag{8}$$

However, when using the substitution $e^x = \cos(u)$, we get $du/dx = -e^x/\sin(u)$ and thus

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{\cos(u)}{\sqrt{1-\cos^2(u)}}\frac{dx}{du}du \tag{9}$$

$$= -\int \frac{\cos(u)}{\sin(u)}\frac{\sin(u)}{e^x}\,du \tag{10}$$

$$= -\int 1\,du = -u + C = -\cos^{-1}(e^x) + C. \tag{11}$$

We then have two values of the integral: $I_1 = \sin^{-1}(e^x)+C$ and $I_2 = -\cos^{-1}(e^x)+C$. However, since

$$\begin{cases} y = \cos(x) \\ y = \sin(\pi/2 - x) \end{cases} \quad \Leftrightarrow \quad \sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2} \tag{12}$$

we see that the difference $I_1 - I_2$ is a constant (Figure 2), hence they are equivalent answers.

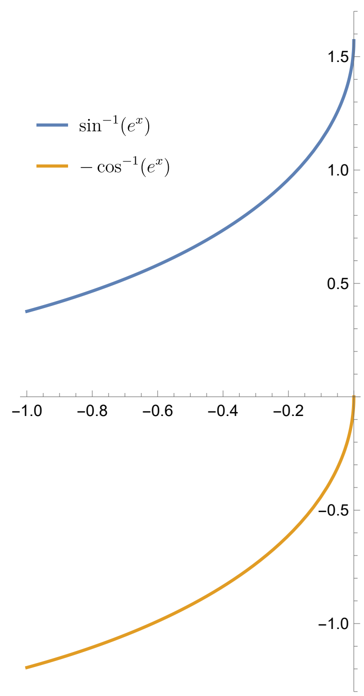

Figure 2

4 Derivation of inverse hyperbolic functions

Hyperbolic sine

Consider the definition of the hyperbolic sine function

$$\sinh(x) = \frac{e^x - e^{-x}}{2}. \tag{13}$$

To find the inverse, set $y = \sinh(x)$ and solve for x

$$y = \frac{e^x - e^{-x}}{2} \Leftrightarrow e^x = y + \sqrt{y^2+1} \Leftrightarrow x = \ln(y + \sqrt{y^2+1}). \tag{14}$$

Therefore, the inverse hyperbolic sine function is

$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1}). \tag{15}$$

Hence, the derivative of $\sinh^{-1}(x)$ is

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{x^2+1}}. \tag{16}$$

Hyperbolic cosine

Consider the definition of the hyperbolic cosine function

$$\cosh(x) = \frac{e^x + e^{-x}}{2}. \tag{17}$$

To find the inverse, set $y = \cosh(x)$ and solve for x

$$y = \frac{e^x + e^{-x}}{2} \Leftrightarrow e^{2x} - 2ye^x + 1 = 0. \tag{18}$$

Solving this quadratic equation for e^x, and taking the positive root since e^x is always positive, leads to

$$e^x = y + \sqrt{y^2-1} \Leftrightarrow x = \ln(y + \sqrt{y^2-1}). \tag{19}$$

Therefore, the inverse hyperbolic cosine function is

$$\cosh^{-1}(x) = \ln(x + \sqrt{x^2-1}). \tag{20}$$

Hence, the derivative of $\cosh^{-1}(x)$ is

$$\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}} \quad \text{for } x > 1. \tag{21}$$

Hyperbolic tangent

Consider the definition of the hyperbolic tangent function

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}. \tag{22}$$

To find the inverse, set $y = \tanh(x)$ and solve for x

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}} \Leftrightarrow e^{2x} - 2ye^x + 1 = 0. \tag{23}$$

Solving for e^x

$$e^x = \frac{1+y}{1-y} \Leftrightarrow x = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right). \tag{24}$$

Therefore, the inverse hyperbolic tangent function is

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right). \tag{25}$$

Hence, the derivative of $\tanh^{-1}(x)$ is

$$\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2} \quad \text{for } -1 < x < 1. \tag{26}$$

5 Alternating series estimation

In Problem 5 (a) from Section I, we are asked to consider the following series, known as the Leibniz formula for π,

$$\pi = 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \tag{27}$$

and estimate how many terms of this series are needed to calculate π to 10 decimal places. Initially, we could think it would be enough to pick the first term of order $n+1$ such that

$$\frac{4}{2(n+1)-1} = \frac{4}{2n+1} < 10^{-10} \tag{28}$$

which holds for $n > 2 \times 10^{10}$. However, some of the following terms might still affect the 10th decimal place. Let's study the general case.

Consider a series of the form

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n, \tag{29}$$

where $\{a_n\}$ is a sequence of positive terms. We have the following theorems:

Theorem 5.1 *If $a_n \geq 0$ for $n = 0, 1, \ldots$ and if the sequence (a_n) decreases monotonically to zero, then the series*

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots \tag{30}$$

converges. Let L be its sum. Moreover, let

$$S_n \coloneqq a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1} a_n, \tag{31}$$

$$R_n \coloneqq L - S_n, \tag{32}$$

denote its nth partial sum and remainder, respectively. Then

$$|R_n| \leq a_{n+1}, \tag{33}$$

and R_n has the sign $(-1)^n$.

Theorem 5.2 *Let $\Delta a_n \coloneqq a_n - a_{n+1}$. If, additionally, the sequence (Δa_n) converges monotonically to zero, then*

$$\frac{a_{n+1}}{2} < |R_n| < \frac{a_n}{2}. \tag{34}$$

Let

$$a_n = \frac{4}{2k-1}. \tag{35}$$

From Theorem 5.2, in order to approximate π to 10 decimal places, we need to determine the minimum k such that

$$\frac{2}{2k+1} < 10^{-10} < \frac{2}{2k-1}, \tag{36}$$

which holds for $k = 10^{10}$. It is important to notice that this only sets a bound to the remainder, and the word *accurate* is important here, as setting this k does not necessarily guarantee that the 10th decimal place is exactly accurate. For a question of this nature, however, an answer based on a bound of the remainder (via either theorem) would suffice.

Without knowing the number we are approximating, we cannot be sure that a certain level of accuracy determines any particular digit accurately (an example of an arbitrarily long string of zeros or nines shows this). In numerical analysis, the standard definition of "accurate to k decimal places" is that the difference between the true value, x, and the approximate value, $\tilde{x}$ have *at least* k zeros in its decimal expansion (regardless of rounding to that decimal place), that is

$$|x - \tilde{x}| \leq 5 \times 10^{-(k+1)}. \tag{37}$$

A refined answer that guarantees this would then be $k = 2 \times 10^{10}$.

The general question of finding the minimum number of terms needed to calculate π accurate to exactly k decimal places is significantly harder. To understand this, we will look at a simpler example.

Example 5.1 (Approximating π) Consider again the Leibniz formula for π

$$\pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}. \tag{38}$$

Now we ask the following question: What is the minimum number of terms needed to calculate estimate π with exactly equal first k decimal places? Notice that this is a different question!

Let's consider $k = 2$ decimal places for example and set $a_n = \frac{4}{2n-1}$. One way to think about this is to consider the remainder of the series and simply use Theorem 5.1

$$|R_n| \leq a_{n+1} \Leftrightarrow |R_n| \leq \frac{4}{2n+1} \leq 10^{-2} \tag{39}$$

which holds for $n \geq 200$. Alternatively, Theorem 5.2 yields

$$\frac{a_{n+1}}{2} < |R_n| < \frac{a_n}{2} \Leftrightarrow \frac{2}{2n+1} < |R_n| < \frac{2}{2n-1} \tag{40}$$

which leads to $99.5 < n < 100.5$ and thus $n = 100$, a refined number. Indeed, either term order satisfies

$$R_{200} = 4 \sum_{n=201}^{\infty} \frac{(-1)^{n-1}}{2n-1} \simeq 0.004999968751 < 10^{-2} \tag{41}$$

$$R_{100} = 4 \sum_{n=101}^{\infty} \frac{(-1)^{n-1}}{2n-1} \simeq 0.009999750031 < 10^{-2} \tag{42}$$

However, the partial sums give

$$S_{200} = 4 \sum_{n=1}^{200} \frac{(-1)^{n-1}}{2n-1} \simeq 3.136592685 \tag{43}$$

$$S_{100} = 4 \sum_{n=1}^{100} \frac{(-1)^{n-1}}{2n-1} \simeq 3.131592904 \tag{44}$$

which are not accurate to two decimal places ($\pi \simeq 3.14...$). In fact, the minimum value of n that gives an accuracy to two decimal places was $n = 119$ (by inspection). Indeed,

$$S_{119} = 4 \sum_{n=1}^{119} \frac{(-1)^{n-1}}{2n-1} \simeq 3.149995867 \tag{45}$$

$$R_{119} = 4 \sum_{n=120}^{\infty} \frac{(-1)^{n-1}}{2n-1} \simeq -0.008403213004 \tag{46}$$

The first 10 minima for "accuracies" $k = 0, ..., 9$ are, respectively,

$$(3, 19, 119, 167, 10794, 136121, 1530012, 18660304, 155973051, 1700659132). \tag{47}$$

6 Exact differentials

Example 6.1 (Percentage errors) The percentage (or *percent*) error of a certain quantity of interest is defined as the difference between a measured or experiment value and an accepted or known value, divided by the known value, multiplied by 100%. As an example, let's consider the volume of a cylinder V given as a function of the radius r and height h (see also Problem 5 from Examples Sheet 2). The percentage error is then given by dV/V. The exact differential for $V \equiv V(r,h)$ is given by

$$dV = V_r\, dr + V_h\, dh \tag{48}$$

where V_r and V_h are the partial derivatives with respect to r and h, respectively. Then, since $V = \pi h r^2$, we have that

$$dV = (2\pi h r)\, dr + (\pi r^2)\, dh. \tag{49}$$

Finally, dividing by V leads to the percentage error

$$\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h}, \tag{50}$$

which is the addition of two percentage errors, corresponding to the errors in measuring r and h. Therefore, the error in a measurement of the volume V can be determined by the independent errors in measurements of either r or h. For example, if we have a 0.1% error in measuring r, and 0.3% in measuring h, we have that the percentage error in measuring V is given by

$$\frac{dV}{V} = 2 \times 0.1\% + 0.3\% = 0.5\%. \tag{51}$$

Naturally, if both errors in measuring r and h are 0%, the error in measuring V is also 0%.

Example 6.2 (Problem 10 from Examples Sheet 2) When showing identities with exact differentials, the most important point is to write everything with respect to the same variables, whether we're dealing with the exact differential form, or in terms of partial derivatives. Here, I will present a detailed solution to Problem 10 by identifying the relevant variables.

First, note that the enthalpy (like entropy) is a *state function* which describes the equilibrium states of a system and implies, therefore, an exact differential. In a system like this, everything is a variable of everything else, so first, note that if U is regarded as a function of p and V ($U \equiv U(p,V)$), then S is also a function of p and V ($S \equiv S(p,V)$), and thus, using the differential form of S, we get

$$dU = T\,dS - p\,dV = T\left(\frac{\partial S}{\partial p}\,dp + \frac{\partial S}{\partial V}\,dV\right) - p\,dV = \left(T\frac{\partial S}{\partial p}\right)dp + \left(T\frac{\partial S}{\partial V} - p\right)dV \tag{52}$$

which is a differential form for U. Hence,

$$\frac{\partial U}{\partial p} = T\frac{\partial S}{\partial p} \tag{53}$$

$$\frac{\partial U}{\partial V} = T\frac{\partial S}{\partial V} - p \tag{54}$$

from which the second cross-derivatives follow

$$\frac{\partial^2 U}{\partial V \partial p} = \frac{\partial}{\partial V}\left(T\frac{\partial S}{\partial p}\right) = \frac{\partial T}{\partial V}\frac{\partial S}{\partial p} + T\frac{\partial^2 S}{\partial V \partial p} \tag{55}$$

$$\frac{\partial^2 U}{\partial p \partial V} = \frac{\partial}{\partial p}\left(T\frac{\partial S}{\partial V} - p\right) = \frac{\partial T}{\partial p}\frac{\partial S}{\partial V} + T\frac{\partial^2 S}{\partial p \partial V} - 1 \tag{56}$$

Since $\frac{\partial^2 U}{\partial V \partial p} = \frac{\partial^2 U}{\partial p \partial V}$ and $\frac{\partial^2 S}{\partial V \partial p} = \frac{\partial^2 S}{\partial p \partial V}$, it follows that

$$\frac{\partial T}{\partial p}\frac{\partial S}{\partial V} - \frac{\partial T}{\partial V}\frac{\partial S}{\partial p} = 1. \tag{57}$$

Example 6.3 (Problem 11 from Examples Sheet 2) Similar to Problem 10, a general approach is to write everything with respect to the same variables. In this case, we want to write dU with respect to the variables involved in G, which are p and T. Hence, we note that

$$dU = T\,dS - p\,dV = d(ST) - S\,dt - d(pV) + V\,dp. \tag{58}$$

Rewriting, and using the properties of differentials, we get

$$d(U + pV - ST) = V\,dp - S\,dT. \tag{59}$$

Hence, $G \equiv U + pV - ST$ (also known as Gibbs free energy).

Other brief notes

- Problem 9.(f) from Examples Sheet 2. Integrating each differential term with respect to different variables leads to

$$f(x,y) = \int \frac{x}{x^2+y^2}\,dy = \tan^{-1}\left(\frac{y}{x}\right) + g_1(x) \tag{60}$$

$$f(x,y) = \int \frac{-y}{x^2+y^2}\,dx = -\tan^{-1}\left(\frac{x}{y}\right) + g_2(y) \tag{61}$$

  where g_1 and g_2 are functions that exclusively depend on x and y, respectively. While the simplest way to determine f would be to differentiate either of the previous forms, one can also notice that

$$\tan^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right) = \operatorname{sgn}(xy)\frac{\pi}{2} \tag{62}$$

  where sgn is the sign function.

- Problem 12.(ii) from Examples Sheet 2. To see why the hint is true, notice that

$$\left(\frac{\partial \ln p}{\partial \ln V}\right)_T = \left(\frac{\partial \ln p}{\partial V}\cdot\frac{\partial V}{\partial \ln V}\right)_T = \left(\frac{1}{p}\frac{\partial p}{\partial V}\cdot V\right)_T = \frac{V}{p}\left(\frac{\partial p}{\partial V}\right)_T. \tag{63}$$