NST1A:
Mathematics II (Course A)

9:00, Tuesday, Thursday Saturday,
Lent 2013

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This version of the notes shows worked examples.

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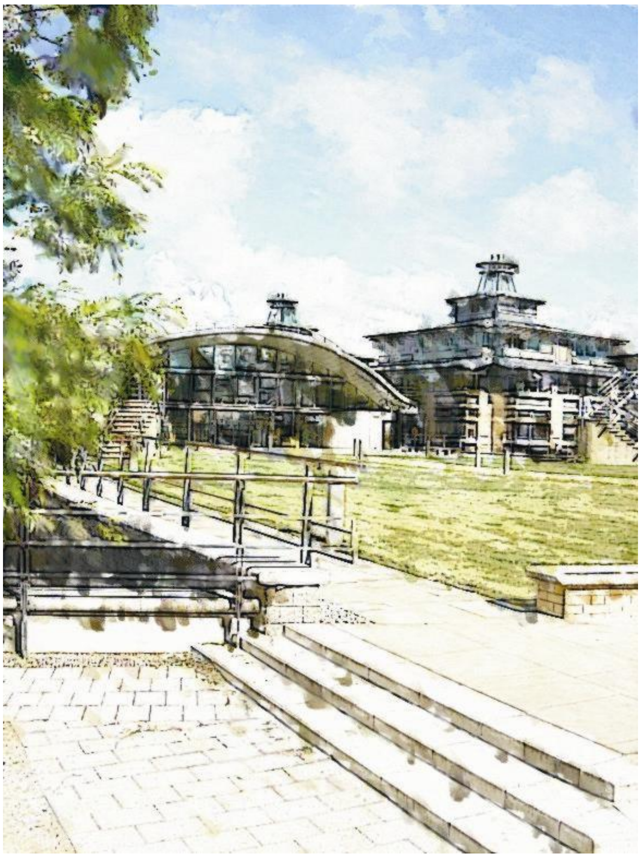

Preamble

These printed handouts are intended to provide a resource to students so that they can concentrate on learning and understanding the mathematical content of the course, rather than just copying down words and equations. The printed handouts do not contain everything of value that will be given in the lectures, and it is expected that students will take their own notes to supplement the handouts. Conversely, the printed notes include some material that is not lectured in detail. Ultimately, the lectures in conjunction with the Schedules determine what is examinable, not the printed notes.

In preparing the first version of these lecture notes (for Lent 2010) I am indebted to Prof. Peter Haynes (the previous lecturer of this course in 2009), from whose notes I have drawn heavily. I have also included material (somewhat modified) from notes I prepared when lecturing the Maths 1A Differential Equations course a few years ago.

Unfortunately, despite my best efforts, some typos have undoubtedly crept into these notes and I would appreciate it if you could advise me of any errors, inconsistencies or lack of clarity you encounter using them.

Stuart Dalziel
January 2013

Course description

The following *course description* can be found at
http://www.maths.cam.ac.uk/undergrad/nst/

The following mathematics courses are provided for Part IA of the Natural Sciences and Computer Science Tripos.

- Mathematics, Course A
- Mathematics, Course B

Course A provides a thorough grounding in methods of mathematical science and contains everything prerequisite for the mathematical content of all physical-science courses in Part IB of the Natural Sciences Tripos, including specifically Mathematics, Physics (Physics A) and Physics (Physics B). Course B contains additional material for those students who find mathematics rewarding in its own right, and it proceeds at a significantly faster pace. Both courses draw on examples from the physical sciences but provide a general mathematical framework by which quantitative ideas can be transferred across disciplines.

Students are strongly encouraged to take Course A unless they have a thorough understanding of material in Further Mathematics A-Level. As a guide, such students might be expected to have scored in the region of 95% in at least two of the modules FP1, FP2, FP3. Some topics that look similar in the Schedules may be lectured quite differently in terms of style and depth. Both courses lead to the same examination and qualification. Mathematics is a skill that requires firm foundations: it is a better preparation for future courses in NST to gain a first-class result having pursued Course A than to gain a second-class result following Course B.

Schedules for Mathematics II (Course A)

In the University, the *Lecture Schedules* provide the ultimate description of the content of a course. The schedules for this course are repeated below, with the numbers in square brackets indicating the approximate number of lectures devoted to each subject.

Ordinary differential equations. First order equations: separable equations; linear equations, integrating factors. Second-order linear equations with constant coefficients; $\exp(\lambda x)$ as trial solution, including degenerate case. Superposition. Particular integrals and complementary functions. Constants of integration and number of necessary boundary/initial conditions. Particular integrals by trial solutions. [6]

Differentiation of functions of several variables. Differentials, chain rule. Exact differentials. Scalar and vector fields. Gradient of a scalar as a vector field. Directional derivatives. Unconditional stationary values. Elementary sketching of contours in two dimensions illustrating maxima, minima and saddle points. Verification of solution to a partial differential equation by substitution. Linear superposition. [7]

Double and triple integrals in Cartesian, spherical and cylindrical coordinates. Examples to include evaluation of $\int_{-\infty}^{\infty} e^{-x^2}\,dx$. [3]

Line integral of a vector field. Conservative and non-conservative vector fields. Surface integrals and flux of a vector field over a surface. Divergence of a vector field. ∇^2 as div grad. Curl. *Statement of the Divergence and Stokes Theorems.* [5]

Extended examples distributed through the course. [3]

Examinations

In the examinations, formulae booklets will not be provided but candidates will not be required to quote elaborate formulae from memory. Calculators are **not** permitted in the Mathematics examinations. (Standard calculators would not provide any benefits for the style of questions set.)

Both Course A and Course B at Part 1A are examined in two three-hour written papers, common to both courses, at the end of the year.

The written papers each consist of two sections, A and B. Section A on Paper 1 is based on the core A-Level syllabus.

All other parts of the written papers are based on these Schedules. Candidates may attempt all questions from Section A and at most 5 questions from section B. Section A on each paper consists of up to 20 short-answer questions and carries a total of 20 marks. Section B on each paper consists of 10 questions, each of which carries 20 marks. Up to 2 of the questions in Section B of each paper are starred to indicate that they rely on material lectured in the B course but not in the A course. The examination paper shows, for each major subsection of a question, the approximate maximum mark available.

The questions in Section A have clear goals that carry 1 mark (correct) or 0 marks (incorrect or incomplete); no fractional credit is given and it is not necessary to show working. In Section B, partial credit may be available for incomplete answers and students are advised to show their working.

Chapters

The material in these printed notes is divided into four 'chapters', reflecting the contents of the Lecture Schedules.

1. Ordinary differential equations

2. Functions of several variables

3. Multiple integration

4. Scalar and vector fields

Formatting

Standard parts of the lecture notes are set in a serif font without a boarder. (Mathematics – including your examinations at the end of the year – is normally typeset with serif fonts.)

Examples

Examples are indicated by a bar down the left-hand side.

Some worked examples are shown in the on-line version of the notes, but not the lecture handouts.

Aside

Remarks that could be useful, but are peripheral to the main idea, are set with a grey background and a thin boarder.

Proofs

These need not be replicated and are set with a grey background and a wiggly boarder. Understanding the proof is not essential for examination purposes of this part of the course.

Advanced

Advanced topics are set with a grey background and a dot-dash border. Understanding this material is not necessary for examination purposes of this part of the course.

Resources

The printed notes and examples sheets that are handed out at the lectures may also be found within CamTools. The electronic version of the printed notes is in colour as a pdf, but the printed version will only be provided in black and white.

Printed copies of the notes can be provided in larger print for any students with visual impairment.

There are **no** examples classes given by the lecturer, but *extended examples* will be included within the lectures.

A text book is **not** required for the course, although some students will benefit from having a text book. The Faculty of Mathematics recommends the following:

†*‡ E Kreyszig
Advanced Engineering Mathematics, 9th edition.
Wiley, 2005 (8th edition 1999)

†* K F Riley, M P Hobson & S J Bence
Mathematical Methods for Physics and Engineering, 3*rd* edition.
Cambridge University Press, 2006 (paperback).

†* G Stephenson
Mathematical Methods for Science Students, 2nd edition.
Prentice Hall/Pearson, 1973 (paperback).

† M L Boas
Mathematical Methods in the Physical Sciences, 3rd edition.
Wiley, 1983 (paperback and hardback)

A Jeffrey
Mathematics for Engineers and Scientists, 6th edition.
Chapman & Hall, 2004 (paperback)

I S Sokolnikoff & R M Redheffer
Mathematics of Physics and Modern Engineering, 2nd edition.
McGraw Hill, 1967 (out of print)

G Stephenson
Worked Examples in Mathematics for Scientists and Engineers.
Longman, 1985 (out of print)

K A Stroud & D Booth
Engineering Mathematics, 6th edition.
Palgrave, 2007 (paperback with CD-ROM)

K A Stroud & D Booth
Advanced Engineering Mathematics.
Palgrave, 2003 (£32.99 paperback)

G Thomas, M Weir, J Hass & F Giordano
Thomas's Calculus, 11th edition.
Pearson, 2004 (paperback; 12th edition in hardback)

† Recommended as principal texts for NST1A Mathematics course

* Particularly recommended for this part of course

‡ Does not do stationary points

1. Ordinary differential equations

1.1 *Introduction*

Scientific problems in *all* fields often lead naturally to mathematical equations relating the derivatives of a function with the value of the function itself. The equations will often depend also on parameters and functions of the independent variable.

1.1.1 Introductory examples

1. Newton cooling

The rate of change of the temperature, θ, of an object is proportional to the difference between the temperature of the object and the temperature of the environment, θ_∞:

$$\frac{d\theta}{dt} = -\alpha\left(\theta - \theta_\infty\right).$$

Here α is a dimensional constant that determines the rate of cooling.

2. Falling mass

A dense object of mass m falling under gravity g through air will accelerate until the drag balances the weight of the mass. If the object is large and moving quickly (so that inertial forces dominate over viscous forces*) then the drag will be proportional to the square of its fall speed v. Newton's laws of motion tell us that

$$mass \times acceleration = force$$

$$\Rightarrow \qquad m\frac{dv}{dt} = mg - a|v|v$$

for some constant a.

* The *Reynolds number* of the flow around the object is said to be large.

3. Moral society

In a model society, we might divide people into two groups: married and single. The evolution of the population requires equations for the number of married people $m(t)$, and the number of single people $s(t)$. If we assume that only married people have children, then a suitable system of equations might be

$$\frac{dm}{dt} = ws - rm,$$

$$\frac{ds}{dt} = -(w + r)s + bm,$$

where w is the net rate of marriages (rate of weddings minus rate of divorces), r is the death rate and b the birth rate.

4. Chemical reaction

Two chemical species A and B react together to form a third species C as

$$A + 2B \rightarrow C.$$

If the concentrations at time t are $a(t)$ = [A], $b(t)$ = [B] and $c(t)$ = [C], then, according to the 'law of mass action',

$$\frac{da}{dt} = -kab^2$$

$$\frac{db}{dt} = -2kab^2$$

$$\frac{dc}{dt} = kab^2$$

where k is the reaction rate.

Common features

These are equations for an unknown function $\theta(t)$ or $v(t)$, or unknown functions $a(t)$, $b(t)$ and $c(t)$.

The equations are 'ordinary differential equations'. They have only one independent variable; here time t.

To complete the solution we require an 'initial condition' (or a condition at some other time) on the function: the initial temperature $\theta(t=0) = \theta_0$, the velocity at time $v(t=t_1) = v_1$, or the initial concentrations $a(0) = a_0$, $b(0) = b_0$ and $c(0) = c_0$.

1.1.2 General form

The above are examples of *differential equations*.

An *ordinary differential equation* (ode) involves derivatives of only one variable. [Later we shall deal with *partial differential equations* where there are derivatives of more than one *independent variable*.]

A general form for a *first-order* ordinary differential is

$$\hat{F}\left(\frac{dy}{dx}, y, x\right) = 0,$$

although in everything we are dealing with we shall assume this can be rearranged to the form

$$\frac{dy}{dx} = F(y, x).$$

Here x is the independent variable, and function $y = y(x)$ specifies the *dependent variable*. This equation is first order as it involves only first order derivatives of y.

The first two examples above are first-order odes.

The third and fourth examples can be treated as systems of *coupled* first-order odes, or as a single first-order ode of the vector function $\mathbf{p}(t) = (m(t), s(t))$ or $\mathbf{a}(t) = (a(t), b(t), c(t))$. [Differential equations for vector functions are outside the scope of this course.] As we shall see later, it is possible to rewrite the third example as a pair of uncoupled second-order equations. The fourth example (which is nonlinear) can likewise be written as uncoupled equations.

A general form for a *second-order* ode is

$$\frac{d^2y}{dx^2} = G\left(\frac{dy}{dx}, y, x\right).$$

For an *n*th order ode this generalises to

$$\frac{d^ny}{dx^n} = H\left(\frac{d^{n-1}y}{dx^{n-1}}, \frac{d^{n-2}y}{dx^{n-2}}, \cdots, \frac{dy}{dx}, y, x\right).$$

The highest derivative in an *n*th order equation is of order *n*. The equation can also contain all the other derivatives, the function itself and the independent variable, but it need not contain all of these.

1.1.3 Review of some basic stuff*

Differentiation as a limit

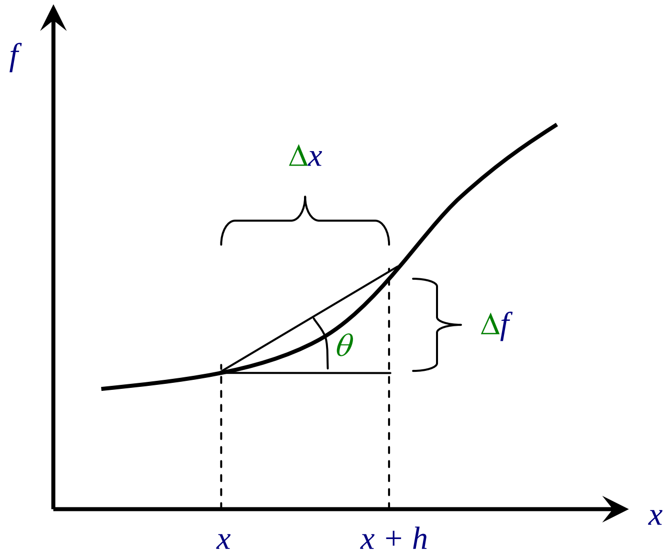

Figure 1: Approximation of slope over finite range of *x*.

Let $$\Delta x \equiv x+h - x = h$$

$$\Delta f \equiv f(x+h) - f(x)$$

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* You should already be comfortable with the ideas in this section. It is included here as a reminder of what you should know, rather than an attempt to teach it to you. You need to be comfortable using the methods for differentiating and integrating functions, rather than knowing details of where they come from.

Define the derivative as

$$\begin{aligned}\frac{df}{dx} &= \lim_{h\to 0}\left(\frac{\Delta f}{h}\right)\\ &= \lim_{\Delta x\to 0}\left(\frac{\Delta f}{\Delta x}\right)\\ &= \lim_{\Delta x\to 0}\left(\tan\theta\right)\\ &= \text{"slope" of curve } f(x)\\ &= \text{"gradient" of curve } f(x)\end{aligned}$$

Other notations

$$\frac{df}{dx},\ \frac{d}{dx}f,\ f',\ f_x,\ f_{,x},\ Df$$

Additionally, especially when $f = f(t)$, the notation $\dot{f} \equiv \frac{df}{dt}$ is often used.

We will use more than one notation, depending on whim, as it is critical you become familiar with the fact the same thing can be expressed in many different ways. The most commonly used notation can vary from one scientific community to another.

> ***Origins of calculus***
>
> In Cambridge we like Newton, but…
>
> Calculus was invented by Sir Isaac Newton and Gottfried Wilhelm Leibniz at around the same time, each claiming to have been first. Leibniz published in 1686 whereas Newton published in 1687, but it appears that Newton actually made the breakthrough some 20 years earlier (1665/66). This led to animosity between them (at least Newton hated Leibniz), and there is/was some suggestion of plagiarism on the part of Leibniz.
>
> Newton's approach was based on the ideas of limits whereas Leibniz used geometric arguments and developed a much simpler notation, including the d/dx and integral symbols we still use. Newton's notation was almost incomprehensible, changing it depending on the context, so was difficult to use and understand. (Was this Newton trying to show off?) British mathematicians used Newton's notation during the 18th century, whereas the rest of the

world adopted Leibniz's notation and made more progress. Some of Newton's notation, for example $\dot{x}$ to represent dx/dt, is still used.

Leonhard Euler adopted the notation Df, D^2f, *etc.*, while his student Joseph-Louis Lagrange developed the still widely used prime notation f', f'', *etc.*

Differentiability

For a function $f(x)$ to be *differentiable* at x:

- the function is continuous
- the derivative f' exists if it is finite and defined.

$\Rightarrow$ the left- and right-hand limits must be the same:

$$\lim_{h\to 0}\left(\frac{f(x)-f(x-h)}{h}\right)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

If f' exists, then $\Delta f \to 0$ as $\Delta x \to 0 \Rightarrow f$ is continuous.

Note: Converse is not necessarily true, *i.e.* continuous f does not necessarily mean f is differentiable.

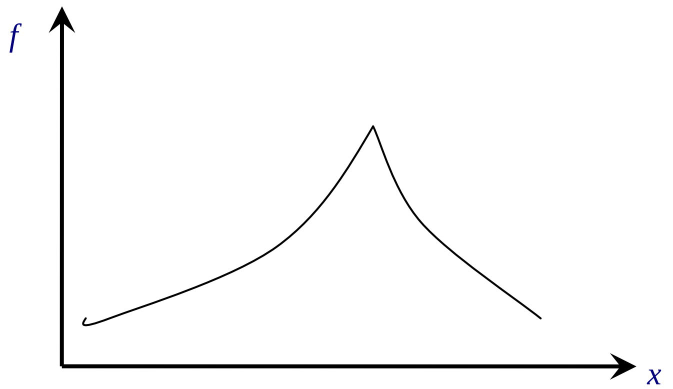


Figure 2: A function with a cusp is continuous, but not differentiable at the cusp as left- and right-hand derivatives are not the same.

The chain rule

Consider $y = f(g(x))$,

$$\frac{d}{dx}f(g(x))=\frac{df}{dg}\frac{dg}{dx}$$

Proof – as a reminder only:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{f(g(x+\Delta x)) - f(g(x))}{g(x+\Delta x) - g(x)} \frac{g(x+\Delta x) - g(x)}{\Delta x} \right)$$

$$= \lim_{\Delta x \to 0} \left(\frac{f(g+\Delta g) - f(g)}{\Delta g} \frac{\Delta g}{\Delta x} \right)$$

$$= \lim_{\Delta g \to 0} \left(\frac{f(g+\Delta g) - f(g)}{\Delta g} \right) \lim_{\Delta x \to 0} \left(\frac{\Delta g}{\Delta x} \right)$$

$$= \frac{df}{dg} \frac{dg}{dx}$$

since $\Delta g \to 0$ as $\Delta x \to 0$ for $g(x)$ (and hence y) to be continuous.

Products

Consider $y = f{\cdot}g$:

The *product rule*: $(f \cdot g)' = f' \cdot g + f \cdot g'$

Proof – as a reminder only:

$$\frac{dy}{dx} = \frac{d}{dx}(f \cdot g) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \left[f(x+h) \frac{g(x+h) - g(x)}{h} + \frac{f(x+h) - f(x)}{h} g(x) \right]$$

$$= f(x) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + g(x) \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= f \frac{dg}{dx} + g \frac{df}{dx}$$

$$= f'g + fg'$$

Example

Consider $y = e^x \cos x$

Let $f(x) = e^x \Rightarrow f' = e^x$

and $g(x) = \cos x \Rightarrow g' = -\sin x$

so $dy/dx = f'g + fg' = e^x(\cos x - \sin x)$.

Quotients

Similarly for quotients $y = f/g$:

Hence the *quotient rule* $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Clearly need $g \neq 0$!

Proof – as a reminder only:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{f}{g}\right) = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{h\,g(x+h)g(x)}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x) - f(x)g(x+h) + f(x)g(x)}{h\,g(x+h)g(x)}$$

$$= \lim_{h\to 0}\left(g(x)\frac{f(x+h)-f(x)}{h\,g(x+h)g(x)}\right) - \lim_{h\to 0}\left(f(x)\frac{g(x+h)-g(x)}{h\,g(x+h)g(x)}\right)$$

$$= \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

$$= \frac{f'g - fg'}{g^2}$$

Example

Consider: $y = \left(\frac{\ln x}{x^2}\right)'$

Let $f(x) = \ln x \Rightarrow f' = 1/x$

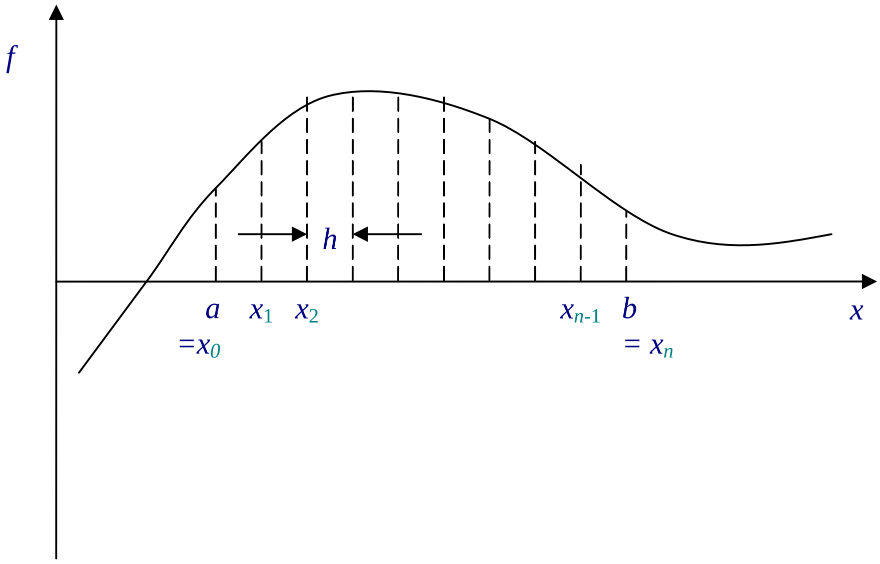
and $g(x) = x^2 \Rightarrow g' = 2x$

so $\frac{dy}{dx} = \frac{f'g - fg'}{g^2} = \frac{\frac{1}{x}x^2 - (\ln x)2x}{(x^2)^2} = \frac{x - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3}$

We could, of course, check this result using the product rule.

Integration

We can consider integration as the limit of a sum.



$$\int_a^b f(x)\,dx \approx I_n = \sum_{i=0}^{n-1} f(x_i)\,h, \text{ where } h = \frac{b-a}{n} = \frac{x_n - x_0}{n}$$

$$\int_a^b f(x)\,dx = \lim_{h\to 0} \sum_i f(x_i)\,h$$

Proof – for interest only:

Let "dx_r" = $x_r - x_{r-1} = h$, and $nh = b - a$, so that the approximate area is the sum of the trapezoids passing through $f_n = f(x_n)$:

$$\int_a^b f(x)\,dx \approx \sum_{r=1}^{n} \tfrac{1}{2}(f_r + f_{r+1})h = \left(\tfrac{1}{2}f_0 + f_1 + f_2 + \cdots + f_{n-1} + \tfrac{1}{2}f_n\right)h$$

This approximation is frequently referred to as the Trapezium Rule and may be used to estimate the integral, for example in a computer code. There are, however, better ways.

If we let $h \to 0$ ($n \to \infty$)

$$\int_a^b f(x)\,dx = \lim_{h\to 0} \sum_{r=1}^{n} \tfrac{1}{2}(f_r + f_{r+1})h$$

$$= \lim_{h\to 0}\left(\tfrac{1}{2}f_0 + f_1 + f_2 + \cdots + f_{n-1} + \tfrac{1}{2}f_n\right)h$$

Since $\lim_{h\to 0} \tfrac{1}{2}(f_0 + f_n)h = 0$, then can rewrite

$$\int_a^b f(x)dx = \lim_{h\to 0}\left(f_0 + f_1 + f_2 + \cdots + f_{n-1} + f_n\right)h$$

$$= \lim_{h\to 0}\sum_i f_i h$$

It is obvious that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

and $$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b \left[f(x) + g(x)\right] dx.$$

Fundamental theorem of calculus

If $f(x)$ is continuous in (a,x), and if $F(x) = \int_a^x f(t)\,dt$, then

$\frac{dF}{dx} = f(x)$ for arbitrary constant a.

Proof (not rigorous) – for interest only:

$$\frac{dF}{dx} = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0}\frac{1}{h}\left(hf(x)\right) = f(x)$$

The function F whose derivative is $f(x)$ is called a *primitive* of f. Note that f will have more than one primitive: if F is a primitive, so is $F + const$.

The primitive is often called the *indefinite integral* and is written as

$$\int^x f(t)\,dt \text{ or more simply as } \int f(x)\,dx$$

because changing a constant lower limit of integration simply adds a constant.

The definite integral specifies both limits:

$$\int_a^b f \; dx = \int_a^b \frac{dF}{dx} \, dx = F(b) - F(a) \equiv F(x)\Big|_a^b \equiv \big[F(x)\big]_a^b$$

Proof – for interest only:

$$\begin{aligned}
\int_a^b f \; dx &= \lim_{h \to 0} h\left(f_0 + f_1 + f_2 + \cdots f_n\right) \\
&= \lim_{h \to 0} h \sum_{k=1}^{n} \frac{F_k - F_{k-1}}{h} \\
&= \sum_{k=1}^{n} \left(F_k - F_{k-1}\right) \\
&= F_n - F_0 \\
&= F(b) - F(a)
\end{aligned}$$

Example

What is dI/dx if $I(x) = \int_{p(x)}^{q(x)} f(t)\,dt$?

If $F(x) = \int_a^x f(t)\,dt$, then $I(x) = F(q(x)) - F(p(x))$, so

$$\begin{aligned}
\frac{dI}{dx} &= \frac{d}{dx}\left[\int_{p(x)}^{q(x)} f(t)\,dt\right] = \frac{d}{dx}\big[F(q(x)) - F(p(x))\big] \\
&= f(q(x))\frac{dq}{dx} - f(p(x))\frac{dp}{dx}
\end{aligned}$$

1.2 First order equations*

The general form of a first-order ode is

$$\frac{dy}{dx} = F(y,x).$$

We cannot, in general, write down a *closed-form solution* of this (*i.e.* $y = Y(x)$). [Often we need to use a computer to find an approximate solution.] However, closed-form solutions are possible for a range of special forms of the function $F(y,x)$.

We shall explore the most common forms and how they can be solved.

1.2.1 Simplest examples of an ode

The simplest example of a first order ode is when we take $F(y,x)$ as a constant.

Example A

Consider $$\frac{dy}{dx} = a.$$

We proceed by integrating both sides with respect to the independent variable x:

$$\int \frac{dy}{dx}\, dx = \int a\, dx.$$

The left-hand side can be dealt with either by recalling the fundamental theorem of calculus that tells us that $y = \int \frac{dy}{dx} dx$, or by noting that the left-hand has the form we use for a change of variable so that

* It is important to be able to use the methods introduced in this section, and to be able to identify what methods are appropriate for a given form of equation. Proofs and derivations are not generally required.

$$\int dy = \int a\,dx.$$

In either case, this readily gives us

$$y = ax + b$$

where b is the arbitrary constant of integration.

Obviously this approach will also work if $F(y,x)$ is just a function of x, the result being $y = \int F(x)\,dx$

A slightly more complex example arises from the equation for Newton cooling, the first example in §1.1.1.

Example B

Solve
$$\frac{d\theta}{dt} = -\alpha\left(\theta - \theta_\infty\right),$$

where α and θ_∞ are constants.

We can obviously rearrange this as

$$\frac{1}{\theta - \theta_\infty}\frac{d\theta}{dt} = -\alpha$$

and integrate both sides

$$\int \frac{1}{\theta - \theta_\infty}\frac{d\theta}{dt}\,dt = \int -\alpha\,dt$$

$$\Rightarrow \qquad \int \frac{1}{\theta - \theta_\infty}\,d\theta = \int -\alpha\,dt$$

$$\Rightarrow \qquad \ln\left|\theta - \theta_\infty\right| = c - \alpha t$$

where c is the constant of integration, so

$$\Rightarrow \qquad \theta = \theta_\infty + Ae^{-\alpha t}$$

where $A = \pm e^c$.

1.2.2 Separable equations

The equation is said to be *separable* if we can separate the parts of $F(y,x)$ that depend on x from the parts of $F(y,x)$ that depend on y. In particular, we need to be able to write $F(y,x) = g(x)/h(y)$ such that

$$\frac{dy}{dx} = F(y,x) = \frac{g(x)}{h(y)}.$$

We can then multiply both sides by $h(y)$ (which might be as simple as $h(y) = const.$) so that

$$h(y)\frac{dy}{dx} = g(x),$$

then integrate both sides with respect to x

$$\int h(y)\frac{dy}{dx}dx = \int g(x)dx.$$

The form of the left-hand side allows simple substitution to convert this from an integral over x to an integral over y:

$$\int h(y)\frac{dy}{dx}dx = \int h(y)dy.*$$

The solution of the ode is therefore simply given by the pair of integrals

$$\int h(y)dy = \int g(x)dx$$

so $$H(y) = G(x) + c$$

* Although this looks like simple cancelling of the dx, it is actually more subtle. We can use the idea of limits to show it works, but will look at this in a different way later (in §2.2.2).

where $G(x)$ and $H(y)$ are the integrals of $g(x)$ and $h(y)$, respectively. Equivalently, we can say that $G'(x) = g(x)$ and $H'(y) = h(y)$. Here c is the arbitrary constant of integration. [We only need one constant although there are two integrals.]

Example A

$$\frac{dy}{dx} - \frac{x+1}{y-1} = 0$$

Begin by rewriting the equation in the standard form:

$$\frac{dy}{dx} = \frac{x+1}{y-1}$$

then multiply both sides by $y - 1$

$$(y-1)\frac{dy}{dx} = x+1$$

Integrate with respect to x, rewriting the left-hand side as an integral with respect to y

$$\int (y-1)\frac{dy}{dx}dx = \int (y-1)dy = \int (x+1)dx$$

$$\Rightarrow \qquad \frac{1}{2}y^2 - y = \frac{1}{2}x^2 + x + c.$$

Note that this is an implicit expression for y. It is often not possible to form an explicit expression for y. Here we have a quadratic for y that we can solve

$$\frac{1}{2}\left(y^2 - 2y + 1\right) - \frac{1}{2} = \frac{1}{2}x^2 + x + c$$

$$(y-1)^2 = x^2 + 2x + 2c + 1$$

Solving the quadratic then gives

$$y = 1 \pm \sqrt{x^2 + 2x + 2c + 1}$$

Which root is relevant for a given x is not clear without the boundary or initial condition; we shall deal with that later.

We can check that this is the solution by substituting back into the equation:

$$\frac{dy}{dx} = \pm\frac{1}{2}\frac{(2x+2)}{\sqrt{x^2+2x+2c+1}}$$

and noting that $y-1=\pm\sqrt{x^2+2x+2c+1}$

gives $$\frac{dy}{dx} = \frac{1}{2}\frac{(2x+2)}{(y-1)} = \frac{x+1}{(y-1)}.$$

Example B

The simplest model for population $p(t)$ growth is that the number of births and deaths are proportional to the population, *i.e.*

$$\frac{dp}{dt} = (b-r)\,p,$$

where b is the birth rate and d the death rate. This equation is almost identical to the Newton cooling equation we have already looked at!

$\Rightarrow$ $$\frac{1}{p}\frac{dp}{dt} = b-r$$

$\Rightarrow$ $$\ln p = (b-r)t + c$$

$\Rightarrow$ $$p = p_0\, e^{(b-r)t},$$

where $p_0 = e^c$ is the initial population at $t = 0$.

Competition

In a more sophisticated model, we might assume that as the population grows the rate of deaths increases due to competition for resources such as food. This can be modelled as

$$\frac{dp}{dt} = bp - rp^2,^*$$

where the quadratic term is indicative of the probability of competition for the resource and both *b* and *r* are constants.

For simplicity we shall take *b* = 1 and *r* = ½.

Rearranging : $$\frac{1}{p\left(1-\frac{1}{2}p\right)}\frac{dp}{dt} = 1$$

$$\Rightarrow \qquad \left(\frac{1}{p} + \frac{1}{2-p}\right)\frac{dp}{dt} = 1$$

and integrate both sides with respect to *t*

$$\int\left(\frac{1}{p} + \frac{1}{2-p}\right)dp = \int dt$$

$$\Rightarrow \qquad \ln p - \ln(2-p) = \ln\frac{p}{2-p} = t + c$$

$$\Rightarrow \qquad \frac{p}{2-p} = Ae^t$$

$$\Rightarrow \qquad p = \frac{2Ae^t}{1+Ae^t} = \frac{2A}{A+e^{-t}}$$

Note that as $t \to \infty$ the population will tend towards *p* = 2 at which point births and deaths are equal.

---

* This equation is also known as the *logistics equation*.

Low population density

If there is a low population density, then the probability of members of the population meeting in order to procreate may be more important than the availability of resources. In such a case a model of the form

$$\frac{dp}{dt} = bp^2 - rp$$

may be more appropriate. This gives

$$p = \frac{r}{b - Be^{rt}} = \frac{re^{-rt}}{be^{-rt} - B}.$$

The sign of B depends on the initial conditions.

The equilibrium population is $p = r/b$, but any departures from this will diverge. Small populations will die out, while large populations will increase without bounds.

Sometimes, we may not be able to find an explicit solution, but have to make do with an implicit one.

Example C

$$\frac{dy}{dx} = y^2(1+y)\cos x$$

$$\Rightarrow \quad \frac{1}{y^2(1+y)}\frac{dy}{dx} = \left(\frac{1}{y^2} - \frac{1}{y} + \frac{1}{1+y}\right)\frac{dy}{dx} = \cos x$$

$$\Rightarrow \quad \int\left(\frac{1}{y^2} - \frac{1}{y} + \frac{1}{1+y}\right)dy = \int \cos x\, dx$$

$$\Rightarrow \quad -\frac{1}{y} - \ln y + \ln(1+y) = \ln\frac{1+y}{y} - \frac{1}{y} = \sin x + c$$

Aside

The notation $\int h(y)\,dy = \int g(x)\,dx$ is potentially confusing. The indefinite integral on the left-hand side should be interpreted as a function of y, while that on the right-hand side should be interpreted as a function of x. We can avoid this confusion by writing instead

$$\int^{y} h(r)\,dr = \int^{x} g(s)\,ds + c\,.$$

There is complete freedom in the naming of the *dummy variables*, here r and s. [It would be confusing to use y or x, but we will often use y' or x', at least when this does not appear like differentiation!]

Since we have replaced indefinite integrals by definite integrals, we must include explicitly the constant of integration c. In both definite integrals we have specified the upper bound (y for the left-hand side and x for the right-hand side), but not the lower bound. Provided the lower bounds are constant then they do not matter: the affect the value of c but not the final expression. The lower bounds for the left- and right-hand sides need not be the same.

1.2.3 General solution and initial conditions

The solution process outlined above has introduced a constant of integration, c. The solution may depend on c in quite complicated ways, with even the form of the solution changing (*e.g.* from an exponential to a sinusoid) depending on the value of c.

The form of the solution containing the constant of integration is referred to as the *general solution*.

To determine a unique solution for a particular problem, we must fix the value of c. The extra piece of information used to do this is often called an *initial condition*. (It can also be called a *boundary condition*.)

Our earlier example (A)

$$\frac{dy}{dx} = \frac{x+1}{y-1}$$

gave
$$\frac{1}{2}y^2 - y = \frac{1}{2}x^2 + x + c.$$

$\Rightarrow$
$$y = 1 \pm \sqrt{x^2 + 2x + 2c + 1}$$

Normally (but not always) we are interested in a unique solution so that either $y = 1 + \sqrt{x^2 + 2x + 2c + 1}$ or $y = 1 - \sqrt{x^2 + 2x + 2c + 1}$.

Suppose we have the initial condition that $y = 0$ on $x = 0$. Hence we require

$$0 = 1 \pm \sqrt{0^2 + 2(0) + 2c + 1}$$

For this to make sense we must select the negative root so that

$$0 = 1 - \sqrt{2c + 1}$$

which requires that $c = 0$.

Hence the specific solution subject to $y(0) = 0$ is

$$\begin{aligned} y &= 1 - \sqrt{x^2 + 2x + 1} \\ &= 1 - \sqrt{(x+1)^2} \\ &= 1 - |x+1| \end{aligned}$$

$\Rightarrow$
$$\begin{aligned} y &= \begin{cases} 1 - (x+1), & x \geq -1 \\ 1 + (x+1), & x < -1 \end{cases} \\ &= \begin{cases} -x, & x \geq -1 \\ x + 2, & x < -1 \end{cases} \end{aligned}.$$

Different boundary conditions

If instead we had the boundary condition $y = 3$ when $x = 0$, then we would have had to take the positive root so that

$$3 = 1 + \sqrt{2c+1}$$

giving $c = 3/2$ and

$$y = 1 + \sqrt{x^2 + 2x + 4}\,.$$

Note that we could have obtained c from the implicit form of the solution:

$$\frac{1}{2}y^2 - y = \frac{1}{2}3^2 - 3 = \frac{3}{2}$$

$$= \frac{1}{2}x^2 + x + c = \frac{1}{2}0^2 + 0 + c$$

Recall our introductory example of a falling mass:

$$m\frac{dv}{dt} = mg - a|v|v\,.$$

If $v \geq 0$, then we may rewrite this as

$$m\frac{dv}{dt} = mg - av^2\,.$$

Rearrange

$$\frac{m}{mg - av^2}\frac{dv}{dt} = \frac{m}{a}\frac{1}{\frac{mg}{a} - v^2}\frac{dv}{dt} = 1\,.$$

To simplify the algebra, let $\beta^2 = mg/a$. We fill find that β is the *terminal velocity* for the mass. Rearrange with partial fractions and integrate so

$$\frac{m}{a}\int \frac{1}{\beta^2 - v^2}\frac{dv}{dt}dt = \frac{m}{2\beta a}\int\left(\frac{1}{\beta - v} + \frac{1}{\beta + v}\right)dv = \int 1 dt$$

$$\Rightarrow \qquad \frac{m}{2\beta a}\ln\frac{\beta + v}{\beta - v} = t + c$$

Take an exponential of both sides

$$\frac{\beta+v}{\beta-v}=\exp\left(\frac{2\beta a}{m}(t+c)\right),$$

where c is a constant.

and solve for v $\beta+v=(\beta-v)\exp(\bullet)$, where $\bullet=\frac{2\beta a}{m}(t+c)$

$$\Rightarrow \qquad v(\exp(\bullet)+1)=\beta(\exp(\bullet)-1)$$

$$\Rightarrow v=\beta\frac{\exp(\bullet)-1}{\exp(\bullet)+1}=\beta\frac{\exp\left(\frac{\bullet}{2}\right)-\exp\left(-\frac{\bullet}{2}\right)}{\exp\left(\frac{\bullet}{2}\right)+\exp\left(-\frac{\bullet}{2}\right)}=\beta\tanh\left(\frac{\beta a}{m}(t+c)\right).$$

To complete the solution, we specify the initial condition that $v(t=0) = v_0$, hence

$$v_0=\beta\tanh\left(\frac{\beta a}{m}c\right) \quad\Rightarrow\quad c=\frac{m}{\beta a}\tanh^{-1}\left(\frac{v_0}{\beta}\right)$$

$$\Rightarrow \qquad v=\beta\tanh\left(\frac{\beta a}{m}t+\tanh^{-1}\left(\frac{v_0}{\beta}\right)\right)$$

1.2.4 Linear equations

A *linear* ordinary differential equation may be written as a linear function of y and its derivatives. In the most general form we might have

$$\psi(x) + \varphi_0(x)\, y(x) + \varphi_1(x)\, y'(x) + \varphi_2(x)\, y''(x) + \ldots = 0.$$

In the case of a *first-order* ordinary differential equation,

$$\frac{dy}{dx}=F(y,x),$$

$F(y,x)$ must be a linear function of y for the ode to be linear. In particular, we must be able to write this as

$$\frac{dy}{dx}+p(x)y=f(x).$$

Sometimes we will choose to write the linear equation as

$$\left[\frac{d}{dx}+p(x)\right]y=+f(x)$$

where $\left[\frac{d}{dx}+p(x)\right]$ is a unary operator that acts on y. It is important to recognise that we can expand this notation, but that the operator $[d/dx + p(x)]$ and function y are not commutative.

Homogeneous equation

The simplest case is when $f(x) = 0$. This simplified form of equation is referred to as the *homogeneous equation* – all terms in the equation involve y or its derivative in a linear manner:

$$\frac{dy}{dx}=-p(x)y.$$

This equation is clearly separable

$$\frac{1}{y}\frac{dy}{dx}=-p(x)$$

$$\Rightarrow \qquad \ln|y|=-\int p(x)\,dx+c.$$

Note the inclusion of the constant c at this point is not essential as we have an indefinite integral. However, we shall proceed with the constant to illustrate its role.

Taking $A = e^c$, we can rewrite this as

$$y=Ae^{-\int p(x)dx}.$$

This is the *general solution*. There is a single (multiplicative) constant of integration, A, that may be determined by matching an initial condition for a specific problem.

If $y(x=0) = y_0$, then the specific solution is obviously just

$$y = y_0 e^{-\int_0^x p(\xi)\,d\xi}$$

Constant coefficients

If $p(x)$ is simply a constant (p_0, say), then $\int p(x)\,dx = p_0 x + c$ and we can write the general solution as

$$y = Ae^{-p_0 x},$$

and use any initial/boundary condition to determine the constant $A = \pm e^{-c}$.

1.2.5 Integrating factors

If $f(x)$ does not vanish in our linear first-order ode, then we have an *inhomogeneous equation*,

$$\frac{dy}{dx} + p(x)y = f(x)^{*},$$

and we need to work a little harder to determine the solution.

We shall handle this equation by seeking a way of transforming it to something that looks like

$$\frac{dJ}{dx} = g(x),$$

an equation that we know how to solve. Here, $J = J(x,y)$. An equation of this form is said to be *exact*.

The transformation we need is surprisingly simple. We begin by choosing some function $I(x)$ and simply multiply our original equation by this:

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* If your equation is not initially in this standard form, start by rewriting it in this form so that the subsequent manipulations always follow the same pattern.

$$I(x)\left[\frac{dy}{dx}+p(x)y\right]=I(x)f(x). \qquad (*)$$

Now, if we chose $J(x,y) = I(x)y$, then we find that

$$\frac{dJ}{dx}=\frac{d}{dx}[Iy]=I\frac{dy}{dx}+y\frac{dI}{dx}.$$

Comparing this with (*), suggests we can write

$$\frac{dJ}{dx}=I(x)\frac{dy}{dx}+\frac{dI}{dx}y \;=\; I(x)\left[\frac{dy}{dx}+p(x)y\right]=I(x)\frac{dy}{dx}+I(x)p(x)y$$
$$=I(x)f(x)$$

provided we choose $I(x)$ as the solution of the homogeneous linear ode

$$\frac{dI}{dx}=I(x)p(x).$$

In particular, we take

$$\frac{1}{I}\frac{dI}{dx}=p$$

$$\Rightarrow \qquad \ln I(x)=\int^x p(\xi)d\xi$$

$$\Rightarrow \qquad I(x)=\exp\left(\int^x p(\xi)d\xi\right)$$

This function, $I(x)$, is referred to as the *Integrating Factor* for the differential equation.

Once we know the integrating factor, we can solve

$$\frac{dJ}{dx}=I(x)f(x),$$

and from $J(x,y) = I(x)y$ determine the behaviour of y.

Finding the solution

We take the equation $\frac{dy}{dx} + p(x)y = f(x)$

and transform it to the exact equation

$$I(x)\left[\frac{dy}{dx} + p(x)y\right] = \frac{d}{dx}\left(I(x)y\right) = I(x)f(x)$$

by finding $I(x) = \exp\left(\int^x p(\xi)\,d\xi\right)$.

The exact equation is then integrated with respect to *x*:

$$\int \frac{d}{dx}\left(I(x)y\right)dx = \int d\left(I(x)y\right) \quad = \quad \int I(x)f(x)\,dx$$

$$\Rightarrow \qquad I(x)y = \int I(x)f(x)\,dx$$

$$\Rightarrow \qquad y = \frac{\int I(x)f(x)\,dx}{I(x)}$$

Writing this in terms of $p(x)$ this is

$$\Rightarrow \qquad y = \frac{\int^x \exp\left(\int^\zeta p(\xi)\,d\xi\right) f(\zeta)\,d\zeta + c}{\exp\left(\int^x p(\xi)\,d\xi\right)},$$

where the dummy variables ξ and ζ have been introduced to avoid ambiguity about what is being integrated.

Parts of the solution

There is only one arbitrary constant in the solution. Any arbitrary constants in the integration associated with forming $I(x)$ either cancel or can be incorporated into c.

The part of the solution containing the arbitrary constant,

$$\frac{c}{I(x)} = \frac{c}{\exp\left(\int^x p(\xi)\,d\xi\right)},$$

is simply the general solution of the homogeneous problem, $dy/dx + py = 0$. This solution to the homogeneous problem is referred to as the *complementary function*.

The other part of the solution, $\dfrac{\int^x \exp\left(\int^\zeta p(\xi)\,d\xi\right) f(\zeta)\,d\zeta}{\exp\left(\int^x p(\xi)\,d\xi\right)}$, is the *particular integral* associated with the nonzero right-hand side $f(x)$. As we shall discuss further later, for linear problems such as this we can simply *add* the particular integral to the *complementary function* to obtain the *general solution*.

In particular, if $y = u$ satisfies $\dfrac{dy}{dx} + p(x)y = f(x)$

and $y = v$ satisfies $\dfrac{dy}{dx} + p(x)y = 0$

then $y = u + A\,v$ satisfies $\dfrac{dy}{dx} + p(x)y = f(x)$, where A is arbitrary.

Proof – must understand idea:

$$\begin{aligned}\frac{d}{dx}(u+Av) + p(u+Av) &= \frac{du}{dx} + pu + A\left(\frac{dv}{dx} + pv\right) \\ &= \frac{du}{dx} + pu = f\end{aligned}.$$

The solution v is called a *complementary function* (CF) because it completes the solution.

Example A

Consider $\dfrac{dy}{dx} = e^{-2x} - 3y$ with $y(0) = 2$.

When working with an equation, it is generally best to start by rewriting it in a familiar form:

$$\frac{dy}{dx} + 3y = e^{-2x}$$

Comparing with the standard form $\dfrac{dy}{dx} + p(x)y = f(x)$ gives

$$p(x) = 3, \quad f(x) = e^{-2x}.$$

The integrating factor is

$$I(x) = \exp\left(\int p(x)dx\right) = \exp\left(\int 3dx\right) = e^{3x}.$$

Multiplying the equation by $I(x)$,

$$I(x)\left(\frac{dy}{dx} + p(x)y\right) = \frac{d}{dx}\left(I(x)y\right) = I(x)f(x)$$

$$e^{3x}\left(\frac{dy}{dx} + 3y\right) = \frac{d}{dx}\left(e^{3x}y\right) = e^{3x} \times e^{-2x} = e^{x},$$

then integrating

$$e^{3x}y = \int e^{x}\,dx = e^{x} + c.$$

The initial condition $y(0) = 2$ requires

$$e^{0}2 = e^{0} + c$$

so $c = 1$ and

$$y = e^{-2x} + e^{-3x}.$$

[Check that this satisfies initial condition: $y = 2 = e^0 + e^0 = 1 + 1$. ✓]

Note that in this case the *particular integral* e^{-2x} is the right-hand side of the original equation. While this will sometimes be the case, it is not always so.

Example B

Consider
$$\frac{dy}{dx} - e^{-\kappa x} + \kappa y = 0$$

Rewriting in standard form
$$\frac{dy}{dx} + \kappa y = e^{-\kappa x}$$

gives the integrating factor $I(x) = e^{+\int p\,dx} = e^{\int \kappa\,dx} = e^{\kappa x}$

$$\Rightarrow \qquad e^{\kappa x}\left[\frac{dy}{dx} + \kappa x\right] = \frac{d}{dx}\left(e^{\kappa x} y\right) = e^{\kappa x}\left(e^{-\kappa x}\right) = 1$$

$$\Rightarrow \qquad e^{\kappa x} y = \int 1\,dx = x + c$$

$$\Rightarrow \qquad y = (x + c)e^{-\kappa x}$$

Here, the particular integral is not the right-hand side of the original equation, but is x times it, *i.e.* $x\,\mathrm{e}^{-\kappa x}$. Note that the *complementary function* is proportional to the right-hand side, so $y = e^{-\kappa x}$ cannot solve the inhomogeneous problem.

Example C

$$\frac{dy}{dt} + ty = t \quad \text{with} \quad y(t_0) = 0.$$

Integrating factor: $I(t) = e^{+\int p\,dt'} = e^{\int t'\,dt'} = e^{\frac{1}{2}t^2}$

$$\Rightarrow \qquad e^{\frac{1}{2}t^2}\left(\frac{dy}{dt} + ty\right) = \frac{d}{dt}\left(e^{\frac{1}{2}t^2} y\right) = te^{\frac{1}{2}t^2}$$

$$\Rightarrow \qquad e^{\frac{1}{2}t^2} y = \int_{t_0}^{t} t' e^{\frac{1}{2}t'^2}\, dt' = \left[e^{\frac{1}{2}t'^2} \right]_{t_0}^{t} = e^{\frac{1}{2}t^2} - e^{\frac{1}{2}t_0^2}$$

$$\Rightarrow \qquad y = 1 - e^{\frac{1}{2}\left(t_0^2 - t^2\right)} = 1 - e^{-\frac{1}{2}\left(t^2 - t_0^2\right)}$$

[Check initial condition: $y = 0 = 1 - \exp(-\frac{1}{2}(t^2 - t_0^2)) = 1 - e^0 = 0$.]

Here $y = 1$ is the particular integral satisfying the right-hand side

$$\frac{dy}{dt} + ty = \frac{d1}{dt} + t = t$$

while $y = \exp(-\frac{1}{2}(t^2 - t_0^2))$ is the complementary function.

1.2.6 Solution by substitution

Many first-order ordinary differential equations are neither separable nor linear. However, in some cases we can find a *substitution* of the form $u = U(x,y)$ that reduces the equation to separable or linear forms.

Homogeneous equations

Suppose $$\frac{dy}{dx} = H\left(\frac{y}{x}\right).$$

This equation is *homogeneous* because it is unchanged if we replace y by λy and x by λx: it looks the same if the (x,y) plane is stretched by the same amount in both directions. The function $H(y/x)$ is homogeneous, whereas the derivative dy/dx is also homogeneous. Every additive term in the equation contains a y or its derivative.

Consider the substitution $u = y/x \quad \Rightarrow \quad y = ux$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}(ux) = u + x\frac{du}{dx}$$
$$= H\left(\frac{y}{x}\right) = H\left(\frac{ux}{x}\right) = H(u)$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{H(u) - u}{x}$$

which is separable and can be solved using the method already discussed. Note that although we may be able to determine an *implicit solution* for y, it might not be possible to write down an explicit one.

Example A

Solve $$\frac{dy}{dx} = \frac{x^2 + y^2}{xy} \quad \text{with} \quad y = 1 \text{ at } x = 1.$$

We must first rewrite the right-hand side to see it is a homogeneous function

$$\frac{x^2 + y^2}{xy} = \frac{x}{y} + \frac{y}{x} = H\left(\frac{y}{x}\right).$$

Using the substitution $y = ux$

$$\frac{dy}{dx} = u + x\frac{du}{dx} = \frac{y}{x} + \frac{x}{y} = u + \frac{1}{u} = H\left(\frac{y}{x}\right) = H(u)$$

$\Rightarrow$ $$x\frac{du}{dx} = \frac{1}{u}$$

$\Rightarrow$ $$u\frac{du}{dx} = \frac{1}{x}$$

$\Rightarrow$ $$\int u\,du = \int \frac{1}{x}dx \;\Rightarrow\; \frac{1}{2}u^2 = \ln|x| + c.$$

The initial condition requires $y = 1$ at $x = 1$, hence $u = y/x = 1$ at $x = 1$ so

$$½ = \ln|1| + c \;\Rightarrow\; c = ½$$

and $$u = \pm\sqrt{2\ln|x| + 1}.$$

We take the positive root in order to satisfy the initial condition $u = 1$.

Finally $$y = ux = x\sqrt{\ln x^2 + 1}.$$

Many, but not all first-order equations can be reduced to separable or linear equations through suitable substitutions. The substitutions are not always obvious!

Exercise: Try Example Sheet 1 Q5 using $u = x + y + 1$.

Bernoulli's differential equation

This equation has the general form

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

and can be reduced to a simpler form by the substitution $z = y^{1-n}$.

Example B

$$\frac{dy}{dx} + xy = x^3y^2.$$

Comparing with the model equation, we have $p(x) = x$, $q(x) = x^3$ and $n = 2$. This suggests using the substitution

$$z = y^{1-2} = 1/y \quad \Rightarrow \quad y = 1/z \quad \Rightarrow \quad dy/dx = -z^{-2}\,dz/dx$$

and

$$-\frac{1}{z^2}\frac{dz}{dx} + \frac{x}{z} = \frac{x^3}{z^2},$$

$\Rightarrow$ $\quad \dfrac{dz}{dx} - xz = -x^3$, which is a linear equation in z.

The integrating factor is $I(x) = \exp\left(\int P\,dx\right) = \exp\left(-\int x\,dx\right) = e^{-x^2/2}$, so

$$\frac{d}{dx}\left(e^{-x^2/2}z\right) = -e^{-x^2/2}x^3.$$

$$\Rightarrow \qquad e^{-x^2/2}z = -\int e^{-x^2/2}x^3\,dx$$

Write $x^3e^{-x^2/2} = x^2 \times xe^{-x^2/2}$ and integrate by parts

$$e^{-x^2/2}z = -\int x^2 \times xe^{-x^2/2}\,dx = \left(x^2+2\right)e^{-x^2/2} + c.$$

$\Rightarrow$ $$z = \left(x^2 + 2\right) + ce^{x^2/2}$$

$\Rightarrow$ $$y = \left[x^2 + 2 + ce^{x^2/2}\right]^{-1}.$$

Exercise: Try Example Sheet 1 Q6a with $z = y^{-4}$ and Q6b with $z = y^{-1}$.

Other substitution strategies

Many more (although by no means all) equations can be solved using substitutions. However, picking the appropriate substitution can be difficult. Some hint can be gained from looking at the form of the equation and looking for recurring themes. The list here gives some examples of what might be worth a try:

$$f\left(x \pm y + c\right)\frac{dy}{dx} \pm f\left(x \pm y + c\right) + g\left(x\right) = 0$$ try $u = x \pm y + c$

$$f\left(x, y\right)\frac{dy}{dx} + g\left(x, y\right) = 0$$ try $u = f(x,y)$

Note that these strategies will not always work. A prerequisite is that you can invert the function so that you can write $y = F(u,x)$. It might be necessary to rearrange the equation first to put it in a more familiar form. Examples Sheet 1 contains some examples where these strategies do work.

Later, in §2.2.6, we will look at another, more methodical method for finding solutions to equations like this. In general, it is best to rule out the simplest approaches first!

1.2.7 First order ode strategy

1. Is equation separable?
2. Is equation linear?
3. Is equation homogeneous? Try $u = y/x$
4. Is equation Bernoulli: $\frac{dy}{dx} + p(x)y = q(x)y^n$? Try $z = y^{1-n}$
5. Is it easier to solve dx/dy than dy/dx?
6. Treat as differential and find integrating factor? (Chapter 2)
7. Look for other substitution? (Examiners will often suggest what you should try.)

1.2.8 Extended first-order examples

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(a) Find the general solutions of the differential equations

$$\text{(i)} \quad (1+x^2)\frac{dy}{dx} = ky,$$

where k is a constant, and

$$\text{(ii)} \quad \frac{dy}{dx} - x^2 y = x^2.$$

(b) By writing $y(x) = x\,v(x)$, or otherwise, solve the differential equation

$$x\frac{dy}{dx} - y = \cot\frac{y}{x},$$

given that $y = \pi/4$ when $x = 1$.

Solution of (a) (i)

The equation is separable: divide by $y(1 + x^2)$ to get

$$\frac{1}{y}\frac{dy}{dx} = \frac{k}{1+x^2}.$$

Integrate $$\int \frac{1}{y}\frac{dy}{dx}dx = \int \frac{1}{y}dy = \int \frac{k}{1+x^2}dx$$

For the right-hand side, use the substitution $x = \tan\theta$ to obtain

$$\ln|y| = k\tan^{-1}x + c.$$

Solving for y gives the general solution:

$$y = C\exp(k\tan^{-1}x),$$

where $C = e^c$ is an arbitrary constant.

Solution of (a) (ii)

The equation,

$$\frac{dy}{dx} - x^2 y = x^2,$$

is a linear equation, and so we may proceed using an integrating factor. However, it is also separable, and this provides the easier route!

$$\frac{dy}{dx} = x^2(1+y)$$

$$\Rightarrow \quad \frac{1}{1+y}\frac{dy}{dx} = x^2$$

$$\Rightarrow \quad \ln|1+y| = \frac{1}{3}x^3 + a$$

$$\Rightarrow \quad 1+y = ce^{\frac{1}{3}x^3}$$

$$\Rightarrow \qquad y = -1 + ce^{\frac{1}{3}x^3}$$

As an inhomogeneous linear equation

Comparing with the general form $dy/dx + py = f$ gives $p = -x^2$ and $f = x^2$. The integrating factor is therefore

$$I(x) = \exp\left(\int p\,dx\right) = \exp\left(-\int x^2\,dx\right) = \exp\left(-\frac{1}{3}x^3\right).$$

Multiplying the equation by $I(x)$

$$I\left(\frac{dy}{dx} - x^2 y\right) = \frac{d}{dx}(Iy) = Ix^2 = x^2 e^{-\frac{1}{3}x^3}.$$

Integrating $\qquad Iy = e^{-\frac{1}{3}x^3} y = \int x^2 e^{-\frac{1}{3}x^3}\,dx = -e^{-\frac{1}{3}x^3} + c,$

where c is an arbitrary constant. Rearranging gives the general solution

$$y = -1 + ce^{\frac{1}{3}x^3}.$$

Solution of (b)

The equation

$$x\frac{dy}{dx} - y = \cot\frac{y}{x}$$

has terms of the form y/x, suggesting it might be a homogeneous equation. The suggested substitution, $y(x) = x\,v(x)$ reinforces this idea.

Writing $y(x) = x\,v(x)$ means $dy/dx = v + x\,dv/dx$. Substituting into equation

$$x\frac{dy}{dx} - y = xv + x^2\frac{dv}{dx} - xv = x^2\frac{dv}{dx}$$

$$= \cot\frac{y}{x} = \cot v$$

This transformed equation is separable

$$\tan v\frac{dv}{dx} = \frac{1}{x^2}.$$

Integrating

$$\int \tan v\,dv = \int \frac{\sin v}{\cos v}dv = -\int \frac{d(\cos v)}{\cos v} = -\ln|\cos v|$$
$$= \int \frac{1}{x^2}dx = -\frac{1}{x} + c$$

The initial condition $y = \pi/4$ when $x = 1$ requires

$$-\ln\left|\cos\frac{\pi}{4}\right| = -\ln\frac{1}{\sqrt{2}} = \frac{1}{2}\ln 2 = -\frac{1}{1} + c,$$

$\Rightarrow$ $c = 1 + ½ \ln 2$.

Substituting

$$\ln|\cos v| = \frac{1}{x} - 1 - \frac{1}{2}\ln 2$$

$\Rightarrow$ $$\cos v = \frac{1}{\sqrt{2}}\exp\left(\frac{1}{x} - 1\right)$$

$\Rightarrow$ $$y = xv = x\cos^{-1}\left[\frac{1}{\sqrt{2}}\exp\left(\frac{1}{x} - 1\right)\right].$$

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Solve the following differential equations:

(a) $$\frac{dy}{dx} = \frac{2x + xy^2}{x^2 y - 3y}$$

such that when $x = 3$, $y = 4$;

(b) $$\frac{dy}{dx}\sin x + y\cos x = \frac{1}{2}\sin 2x$$

such that when $x = \pi/6$, $y = ¼$;

(c) $$\left(2ye^{y/x} + x\right)\frac{dy}{dx} - 2x - y = 0.$$

Solution of (a)

The right-hand side of this equation can be factorised:

$$\frac{dy}{dx} = \frac{2x + xy^2}{x^2 y - 3y} = \frac{x\left(2 + y^2\right)}{\left(x^2 - 3\right)y},$$

showing it is separable as

$$\frac{y}{y^2 + 2}\frac{dy}{dx} = \frac{x}{x^2 - 3}.$$

Integrating

$$\int \frac{y}{y^2 + 2}\,dy = \frac{1}{2}\ln\left(y^2 + 2\right)$$

$$= \int \frac{x}{x^2 - 3}\,dx = \frac{1}{2}\ln\left(x^2 - 3\right) + c$$

Simplifying $$y^2 + 2 = C\left(x^2 - 3\right).$$

The initial condition $y(3) = 4$ requires

$$4^2 + 2 = 18$$

$$= C\left(3^2 - 3\right) = 6C$$

$\Rightarrow$ $$C = 3,$$

thus $$y^2 + 2 = 3\left(x^2 - 3\right)$$

$\Rightarrow$ $$y^2 = 3x^2 - 11.$$

The initial condition $y(3) = 4 > 0$ requires we take the positive root, so

$$y = \sqrt{3x^2 - 11}.$$

Solution of (b)

$$\frac{dy}{dx}\sin x + y\cos x = \frac{1}{2}\sin 2x$$

Rewrite in standard form
$$\frac{dy}{dx} + y\frac{\cos x}{\sin x} = \frac{dy}{dx} + y\cot x$$
$$= \frac{1}{2}\frac{\sin 2x}{\sin x} = \cos x$$

This is a linear equation that we may solve by introducing an integrating factor

$$I(x) = \exp\left(\int p\,dx\right) = \exp\left(\int \frac{\cos x}{\sin x}dx\right)$$
$$= \exp\left(\int \frac{d(\sin x)}{\sin x}\right) = \exp\left(\ln(\sin x)\right) .$$
$$= \sin x$$

Multiplying the ode by the integrating factor

$$I\left(\frac{dy}{dx} + y\frac{\cos x}{\sin x}\right) = \frac{d}{dx}(Iy)$$
$$= I\cos x = \sin x\cos x = \frac{1}{2}\sin 2x$$

and integrate

$$Iy = y\sin x$$
$$= \int \frac{1}{2}\sin 2x\,dx = -\frac{1}{4}\cos 2x + \hat{c} = -\frac{1}{2}\cos^2 x + \tilde{c} = \frac{1}{2}\sin^2 x + c$$

Could have noted that $\int \cos x \sin x\,dx = \int \sin x\,d(\sin x) = \frac{1}{2}\sin^2 x$

$$\Rightarrow \qquad y = \frac{1}{2}\sin x + \frac{c}{\sin x}.$$

Taking initial condition $y(\pi/6) = ¼$ requires

$$\frac{1}{4}=\frac{1}{2}\sin\frac{\pi}{6}+\frac{c}{\sin\frac{\pi}{6}}=\frac{1}{2}\frac{1}{2}+\frac{c}{1/2}=\frac{1}{4}+2c,$$

hence $c = 0$ and $y = ½ \sin x$.

Had we been more awake we might have noticed that the left-hand side of the original equation

$$\frac{dy}{dx}\sin x+y\cos x=\frac{d}{dx}\left(y\sin x\right),$$

so we could have missed out the step of computing the integrating factor and integrated the equation directly:

$$y\sin x=\int\tfrac{1}{2}\sin 2x\,dx.$$

Solution to (c)

$$\left(2ye^{y/x}+x\right)\frac{dy}{dx}-2x-y=0$$

The solution of this one is less obvious as the equation is neither linear nor separable. Instead we must seek a substitution.

The presence of y/x in the exponential term gives the hint that $u = y/x$ might be useful, particularly if we recognise that we can divide through by x so that

$$\left(2\frac{y}{x}e^{y/x}+1\right)\frac{dy}{dx}-2-\frac{y}{x}=0.$$

Define $y = ux$, then $dy/dx = x\,du/dx + u$

$$\left(2ue^u+1\right)\left(x\frac{du}{dx}+u\right)-2-u=x\left(2ue^u+1\right)\frac{du}{dx}+2u^2e^u+u-2-u$$

$$\Rightarrow \qquad =x\left(2ue^u+1\right)\frac{du}{dx}+2\left(u^2e^u-1\right)$$

$$=0$$

This equation is now separable

$$\frac{2ue^u+1}{u^2e^u-1}\frac{du}{dx}=-\frac{2}{x}$$

and can be integrated

$$\int\frac{2ue^u+1}{u^2e^u-1}du=-\int\frac{2}{x}dx$$

This is easier if we separate the *u* from the exponential on the left-hand side

$$\int\frac{2u+e^{-u}}{u^2-e^{-u}}du=-\int\frac{2}{x}dx,$$

as it is then obvious that the numerator is the derivative of the denominator so

$$\int\frac{2u+e^{-u}}{u^2-e^{-u}}du=\ln\left|u^2-e^{-u}\right|$$

$$=-\int\frac{2}{x}dx=-2\ln|x|+c=\ln\frac{C}{x^2}$$

Simplifying

$$\left|u^2-e^{-u}\right|=\frac{C}{x^2}$$

and recalling that $y = ux$

$$\left|\left(\frac{y}{x}\right)^2-e^{-y/x}\right|=\frac{C}{x^2}$$

$$\Rightarrow \qquad y^2-x^2e^{-y/x}=\hat{C}.$$

[We cannot write an explicit form for *y*.]

1.3 Second order equations*

A general form of a second-order ode is

$$\frac{d^2y}{dx^2} = F\left(\frac{dy}{dx}, y, x\right).$$

In this section we shall focus on *linear equations*.

We can view the process of reducing the d^2y/dx^2 term to y as integrating twice. Each time we integrate we must introduce an arbitrary constant. Thus, the solution of a second order equation will have *two* arbitrary constants.

1.3.1 Linear equations

As noted previously, a differential equation is *linear* if it can be written in a form where the coefficients multiplying the function y and all its derivatives do not themselves depend on y or its derivatives.

The most general form for a second-order linear ordinary differential equation is

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x).$$

Consequences of linearity

Suppose $y = u$ is the *particular integral*, a solution of the full inhomogeneous equation, *i.e.* $\frac{d^2u}{dx^2} + p(x)\frac{du}{dx} + q(x)u = f(x)$

and $y = v$ is a *complementary function*, a solution of the corresponding homogeneous equation*, *i.e.*

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* In this section we concentrate on linear equations. Again, being able to apply the tools introduced here is important; understanding why they work is not.

$$\frac{d^2v}{dx^2}+p(x)\frac{dv}{dx}+q(x)v=0,$$

then $y = u + Cv$ is also a solution of the original equation. Here C is an arbitrary constant.

This ability to add multiples of the complementary function(s) to the particular integral is referred to as the *principle of linear superposition*.

Proof is straightforward. Let $y = u + Cv$, then

$$\begin{aligned}\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y &= \left[\frac{d^2u}{dx^2}+p(x)\frac{du}{dx}+q(x)u\right]\\ &+C\left[\frac{d^2v}{dx^2}+p(x)\frac{dv}{dx}+q(x)v\right]\\ &= f(x)+C\times 0=f(x)\end{aligned}$$

As we shall see shortly, a second-order linear ode has two linearly independent complementary functions (solutions of the corresponding homogeneous equation). Suppose v and w are linearly independent solutions of the homogeneous equation, then any linear combination of these with the particular integral (u) is a solution of the original equation. In particular,

$$y = u + Cv + Dw,$$

with C and D arbitrary constants, is the most general solution of the original equation.

This principle of linear superposition greatly simplifies the identification of solutions to a linear equation.

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* The term ‘homogeneous’ has many different meanings depending on the context. A linear equation with $f(x) = 0$ is referred to as homogeneous, but this is quite distinct from an equation of the form $dy/dx = H(y/x)$, which is also referred to as homogenous. For the linear equation, ‘homogeneous’ means that y or its derivatives are in every term, whereas for the second case, the function $H(y/x)$ is homogeneous (scale invariant).

1.3.2 Linear equations with constant coefficients

First order equation

When we were looking at the first-order, linear, homogeneous equation with constant coefficients,

$$\frac{dy}{dx} + py = 0$$

we found that the solution was $y = A\,e^{-px}$.

We could have obtained this in a different way by assuming a solution of the form, $y = e^{\lambda x}$ and substituting this into the differential equation. We would then have

$$\frac{dy}{dx} + py = \lambda e^{\lambda x} + pe^{\lambda x} = 0.$$

Dividing through by $e^{\lambda x}$ then leaves us with the *auxiliary equation* (sometimes referred to as the *characteristic equation*)

$$\lambda + p = 0,$$

the solution of which gives $\lambda = -p$, and hence $y = e^{-px}$, as expected.

Second order equation

The case where $p(x)$ and $q(x)$ are constants is very important as well as greatly simplifying the solution procedure. In particular, we are interested in equations of the form

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f(x).$$

We shall begin by considering the homogeneous form of this equation,

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0,$$

in order to determine the complementary functions.

Trial solution

By analogy with the first order equation $dy/dx + py = 0$, the function $y = e^{\lambda x}$ is a candidate for solving the second order equation. Substituting

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = \frac{d^2e^{\lambda x}}{dx^2} + p\frac{de^{\lambda x}}{dx} + qe^{\lambda x}$$
$$= \lambda^2 e^{\lambda x} + p\lambda e^{\lambda x} + qe^{\lambda x}$$
$$= \left(\lambda^2 + p\lambda + q\right)e^{\lambda x} = 0$$

Auxiliary equation

We can therefore see that if λ is a root of

$$\lambda^2 + p\lambda + q = 0,$$

then $e^{\lambda x}$ is a solution of the homogeneous equation. This equation for λ is referred to as the *auxiliary equation*, or sometimes the *characteristic equation*.

There are always two roots of this auxiliary equation:

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

For convenience, we shall label these roots λ_1 and λ_2.

Both $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are solutions to the homogeneous equation. Moreover, provided $\lambda_1 \neq \lambda_2$, the solutions are linearly independent. [We shall deal with the case $\lambda_1 = \lambda_2$ later.] Thus, $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are the *complementary functions* of the equation and

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

is a solution to the homogeneous equation for arbitrary constants A and B. This is the (most) general form of the solution.

Complex roots

The roots of the auxiliary equation,

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2},$$

will be real if $p^2 \geq 4q$. However, if $p^2 < 4q$, then λ_1 and λ_2 form a complex conjugate pair. The general form of the solution

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

is still valid, and we note that the constants A and B may themselves be complex. However, it is often more informative to rewrite the solution using trigonometric functions:

$$y = e^{-\frac{1}{2}px}\left[\hat{A}\cos\left(\sqrt{q - \tfrac{1}{4}p^2}\,x\right) + \hat{B}\sin\left(\sqrt{q - \tfrac{1}{4}p^2}\,x\right)\right].$$

Here, $\hat{A}$ and $\hat{B}$ are again arbitrary constants.

Other notations

Note that it is quite common for us to write the second-order equation in other forms. For example,

$$y'' + py' + qy = f(x),$$

$$\ddot{y} + p\dot{y} + qy = f(t),$$

$$\left[\frac{d^2}{dx^2} + p\frac{d}{dx} + q\right]y = f(x),$$

where we treat the expression in square brackets in the last of these as a unary *operator* that here acts on y. You will find some examples phrased in this way on the first Examples Sheet.

We might also sometimes see the operator factorised in the form

$$\left[\frac{d}{dx}+a\right]\left[\frac{d}{dx}+b\right]y.$$

This may readily be expanded in a manner similar to a quadratic:

$$\begin{aligned}\left[\frac{d}{dx}+a\right]\left[\frac{d}{dx}+b\right]y &= \frac{d}{dx}\left[\frac{d}{dx}+b\right]y+a\left[\frac{d}{dx}+b\right]y \\ &= \frac{d^2y}{dx^2}+(b+a)\frac{dy}{dx}+aby \\ &= \left[\frac{d^2}{dx^2}+(a+b)\frac{d}{dx}+ab\right]y\end{aligned}$$

The auxiliary equation for this is obviously

$$\lambda^2+(a+b)\lambda+ab=(\lambda+a)(\lambda+b)=0,$$

giving solutions $\lambda = -a$ and $\lambda = -b$. Note that these solutions are obvious from the initial factorised form of the operator!

1.3.3 Initial conditions and boundary conditions

For the first-order equations, the general solution contained a single arbitrary constant, the result of a single integration. With linear equations, we had a single complementary function with its arbitrary constant of integration. In all cases, to obtain the specific solution for a given problem, we had to make use of a single initial (or boundary) condition to determine the arbitrary constant.

The picture for second-order equations is similar, except that we have to integrate *twice* to remove the highest order derivative. As we have seen, for linear equations this means the general solution contains *two* complementary functions, each with its own arbitrary multiplicative constant. In order to determine the specific solution we need *two* pieces of information to determine the appropriate values of the arbitrary constants.

The information used to determine the arbitrary constants can arise from *initial conditions* or *boundary conditions*.

We generally use the term 'initial conditions' to indicate that all the information we require is available at a single value of the independent variable. When the independent variable represents time, it is natural (but not necessary) for us to know all the information required at the *start* of the problem. With a second-order problem, initial conditions give us information on both y and dy/dx at a given x.

In a problem where *boundary conditions* are specified, we use information about the solution from two different locations of x. For example, if the boundaries are at $x = x_0$ and $x = x_1$, we might know $y(x_0)$ and $y(x_1)$. Alternatively, we might know $y(x_0)$ and dy/dx at $x = x_1$. Other combinations are also possible, but some care is needed to ensure the two pieces of information are independent.

Sometimes we use the *terms initial value problem* or *boundary value problem* to distinguish the two cases.

Example: Mass on a spring

The oscillation of a mass suspended by a spring leads to a second order system. If we neglect friction for the moment, then Newton tells us that $F = ma$. Here the force F varies due to the stretching of the spring as Kz, where z is the position of the mass m and K is the spring constant. The force is obviously directed in the opposite direction to z. The acceleration a is simply d^2z/dt^2, so

$$-Kz = m\frac{d^2z}{dt^2}.$$

Rewriting this in standard form

$$\frac{d^2z}{dt^2} + \frac{K}{m}z = 0,$$

leads to the auxiliary equation

$$\lambda^2 + \frac{K}{m} = 0.$$

The roots of this give complex $\lambda = \pm i\sqrt{\frac{K}{m}}$, so the general solution is

$$z = A\cos(\omega t) + B\sin(\omega t)$$

where the angular frequency ω is given by $\omega^2 = K/m$. [Note that ω has units of *radians* per unit time.]

Initial conditions

We might have initial conditions for this problem. For example, at $t = 0$ we might know the displacement $z = 1$ and the speed $dz/dt = 1$.

The first of these conditions tells us that $A = 1$, but does not restrain B.

Differentiating the general solution gives us the velocity of the mass as

$$\frac{dz}{dt} = -\omega A\sin(\omega t) + \omega B\cos(\omega t).$$

The condition $dz/dt = 1$ at t=0 then requires $\omega B = 1$. Thus the specific solution is

$$\begin{aligned} z &= \cos(\omega t) + \frac{1}{\omega}\sin(\omega t) \\ &= \cos\left(\sqrt{\frac{K}{m}}t\right) + \sqrt{\frac{m}{K}}\sin\left(\sqrt{\frac{K}{m}}t\right) \end{aligned}$$

Note that if the initial conditions had been at some other time t_0, then they would have given rise to a pair of simultaneous algebraic equations for A and B.

Boundary conditions

Instead of initial conditions, we might have known that the mass was at $z = z_0$ at time $t = t_0$ and at $z = z_1$ at time $t = t_1$.

t_0: $A\cos(\omega t_0) + B\sin(\omega t_0) = z_0$

t_1: $A\cos(\omega t_1) + B\sin(\omega t_1) = z_1$

Thus we have a pair of simultaneous equations to solve:

$$A\,[\cos(\omega t_0)\sin(\omega t_1) - \cos(\omega t_1)\sin(\omega t_0)] = z_0\sin(\omega t_1) - z_1\sin(\omega t_0)$$

$$B[\cos(\omega t_1)\sin(\omega t_0) - \cos(\omega t_0)\sin(\omega t_1)] = z_0\cos(\omega t_1) - z_1\cos(\omega t_0)$$

… *etc*.

$$\Rightarrow \qquad A = \frac{z_0\sin(\omega t_1) - z_1\sin(\omega t_0)}{\cos(\omega t_0)\sin(\omega t_1) - \cos(\omega t_1)\sin(\omega t_0)}$$

$$B = -\frac{z_0\cos(\omega t_1) - z_1\cos(\omega t_0)}{\cos(\omega t_0)\sin(\omega t_1) - \cos(\omega t_1)\sin(\omega t_0)}$$

This solution seems fine, but we need to be careful. This is most obvious if we use trig identities to rewrite these constants as

$$A = \frac{z_0\sin(\omega t_1) - z_1\sin(\omega t_0)}{\sin(\omega(t_1 - t_0))}, \quad B = -\frac{z_0\cos(\omega t_1) - z_1\cos(\omega t_0)}{\sin(\omega(t_1 - t_0))}.$$

Obviously the denominator vanishes if t_0 and t_1 differ by an integer number of half periods π/ω.

If $t_1 - t_2 = n\pi/\omega$ (for integer n) then one of two things can happen:

(a) If $z_0 = z_1$ then the solution is underspecified and there are an infinite number of solutions that satisfy the boundary conditions. The solution is not unique; the problem is not *well posed*. [Detailed discussion of the requirements for a solution to be *well posed* solution is beyond the scope of this course.]

(b) If $z_0 \neq z_1$ then the solution does not exist: we require it to have two different values at the same point. Again, the solution is not well posed.

Boundary conditions (2)

Instead of specifying z at two times, we might specify z at one time and dz/dt at another. For example, if $z = 0$ at $t = 0$ and $dz/dt = 1$ at $t = 1$, then

$t = 0$: $$z = A\cos(\omega t) + B\sin(\omega t) = A = 0$$

$t = 1$: $$\frac{dz}{dt} = \omega B\cos(\omega t) = \omega B\cos\omega = 1$$

$\Rightarrow$ $$A = 0,\ \ B = 1/(\omega\cos\omega).$$

While this solution will normally be well posed, there will be cases (*e.g.* if $\omega = (2n+1)\pi/2$) when the boundary conditions cease to be mutually compatible.

Boundary conditions (3)

It is also, of course, possible to specify the velocity dz/dt at two separate times. For example, if at $dz/dt = 0$ at $t = 0$ and $dz/dt = 1$ at $t = 1$, then

$$t = 0:\ \frac{dz}{dt} = -\omega A\sin(\omega t) + \omega B\cos(\omega t) = \omega B = 0 \ \Rightarrow\ B = 0$$

$$t = 1:\ \frac{dz}{dt} = -\omega A\sin(\omega t) = -\omega A\sin\omega = 1 \ \Rightarrow\ A = -1/(\omega\sin\omega)$$

Again we must be careful about where we specify the boundary conditions as the above solution ceases to make sense if $\omega = n\pi$.

1.3.4 Degenerate case: repeated root

Returning to our general second-order linear equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f(x),$$

the corresponding auxiliary equation

$$\lambda^2 + p\lambda + q = 0$$

has a repeated root when $p^2 = 4q$. Although this suggests there are two identical complementary functions, but we need two *different* complementary functions: by 'integrating' d^2y/dx^2 twice to determine y, we **must** introduce two arbitrary constants of integration, and so our general solution must have two arbitrary constants, not just one. The complementary functions must be *linearly independent*.

Example: a simple repeated root

Consider
$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0.$$

$$\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0$$

with the solution $\lambda = -1$ twice. Following what we did earlier, this suggests a general solution

$$y = Ae^{-x} + Be^{-x} = (A+B)e^{-x} = Ce^{-x}$$

which has only one arbitrary constant!

We must seek elsewhere for a second complementary function.

Up until now we have determined the complementary functions by trying solutions of the form $e^{\lambda x}$. However, when there is a repeated root, we find the complementary functions are functions are $e^{\lambda x}$ and $xe^{\lambda x}$.

Example: Factorising a repeated root

Consider $$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$$

which may be written as

$$\left[\frac{d^2}{dx^2}+2\frac{d}{dx}+1\right]y=0.$$

The auxiliary equation is

$$\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0.$$

The factorisation suggests we can do a similar thing with the differential operator:

$$\left[\frac{d^2}{dx^2}+2\frac{d}{dx}+1\right]=\left[\frac{d}{dx}+1\right]\left[\frac{d}{dx}+1\right]=\left[\frac{d}{dx}+1\right]^2$$

and so write the equation as

$$\left[\frac{d}{dx}+1\right]\left[\frac{d}{dx}+1\right]y=0.$$

If we let $\zeta=\left[\frac{d}{dx}+1\right]y$, then we can write a first order linear equation for ζ as

$$\left[\frac{d}{dx}+1\right]\zeta=0 \quad\Rightarrow\quad \frac{1}{\zeta}\frac{d\zeta}{dx}=-1$$

$\Rightarrow$ $$\zeta = Ae^{-x}.$$

Substituting this into the relationship between ζ and y gives the inhomogeous linear equation

$$\zeta=\left[\frac{d}{dx}+1\right]y=\frac{dy}{dx}+y=Ae^{-x}.$$

The integrating factor is $I=\exp\left(\int 1\,dx\right)=e^x$ and

$$\frac{d}{dx}(Iy)=Ie^{-x}=A$$

$$\Rightarrow \qquad Iy = Ax + B$$

$$\Rightarrow \qquad y = (Ax + B)e^{-x}$$

In general, when $p^2 = 4q$ the auxiliary equation may be rewritten as

$$\lambda^2 + p\lambda + q = \lambda^2 + p\lambda + \frac{1}{4}p^2 = \left(\lambda + \frac{1}{2}p\right)^2 = 0,$$

but rather than factorising the operator, it is simpler to simply remember that the complementary functions will be $e^{\lambda x}$ and $xe^{\lambda x}$.

Advanced – another way of looking at it (not examinable)

Substituting $y = xe^{\lambda x}$ into the homogeneous equation gives

$$\begin{aligned}\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy &= \frac{d^2}{dx^2}\left(xe^{\lambda x}\right) + p\frac{d}{dx}\left(xe^{\lambda x}\right) + qxe^{\lambda x} \\ &= \frac{d}{dx}\left(e^{\lambda x} + \lambda xe^{\lambda x}\right) + p\left(e^{\lambda x} + \lambda xe^{\lambda x}\right) + qxe^{\lambda x} \\ &= \left(\lambda e^{\lambda x} + \lambda e^{\lambda x} + \lambda^2 xe^{\lambda x}\right) + p\left(e^{\lambda x} + \lambda xe^{\lambda x}\right) + qxe^{\lambda x} \\ &= \left(\lambda^2 + p\lambda + q\right)xe^{\lambda x} + (2\lambda + p)e^{\lambda x}\end{aligned}$$

For $xe^{\lambda x}$ to be a complementary function, the coefficient $(\lambda^2 + p\lambda + q)$ for the $xe^{\lambda x}$ term must vanish. This condition is satisfied provided λ is a solution of the auxiliary equation.

However, we also require that $2\lambda + p$ vanishes for the $e^{\lambda x}$ term to vanish. This is exactly the condition for the auxiliary equation to have a repeated root. Specifically, the roots of the auxiliary equation

$$\lambda = -\frac{p}{2} \pm \frac{1}{2}\sqrt{p^2 - 4q}$$

are both $\lambda = -p/2$ when $p^2 = 4q$.

1.3.5 Finding a particular integral

For a homogeneous equation, all we need is the complementary functions. However, if the equation is inhomogeneous,

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=f(x),$$

we also need the *particular integral* associated with the right-hand side $f(x)$.

First-order equations

For first-order equations we had a systematic approach (using an integrating factor) to find the particular integrals. However, in some cases it would have been possible (and faster!) to ‘guess’ the solution and then check it was correct by substitution. For example, with

$$\frac{dy}{dx}+y=1$$

we might try $y = ax + b$. Substituting gives

$$a + ax + b = 1$$

and equating terms shows that $a = 0$ and $b = 1$ so that $y = 1$ is the particular integral, thus the general solution is

$$y = Ae^{-x} + 1.$$

Second-order equations

There is a general integral expression for a particular integral for second-order equations in terms of the function $f(x)$ and the two complementary functions (similar to the way the integrating factor was related to the complementary function for a first-order ode), but this is relatively complicated and rarely used or remembered (or taught).

Instead, we normally inspect the form of $f(x)$ and recognise that certain simple forms of $f(x)$ imply corresponding simple forms of the particular integral. The linearity of the equation helps with this process, as we can add them up if $f(x)$ contains a number of terms added together.

Trial solution

As with the first order inhomogeneous equation, we can pose *trial solutions* for the particular integral of the second order inhomogeneous equation.

f(x) = const

Consider $$y'' + y = 1$$

The homogeneous equation gives rise to the auxiliary equation $\lambda^2 + 1 = 0$, yielding $\lambda = \pm i$ and complementary functions $\sin x$ and $\cos x$.

Consider the trial solution $y = a + bx + cx^2$, and substitute:

$$y'' + y = 2c + (a + bx + cx^2) = 1$$

As this must hold for all x, we have the particular integral is $y = 1$, and the general solution is

$$y = A \sin x + B \cos x + 1.$$

Clearly, introducing any other powers of x to the trial solution would demonstrate they had zero coefficients.

Polynomial f(x)

Consider $y'' - 3y' - y = x^2 + 2$

The auxiliary equation $\lambda^2 - 3\lambda - 1 = 0$ gives $\lambda = (3 \pm \sqrt{13})/2$ and exponential complementary functions.

Try $y = a + bx + cx^2$, and substitute

$$2c - 3(b + 2cx) - (a + bx + cx^2) = (2c - a - 3b) + (-6c - b)x + (-c)x^2$$
$$= x^2 + 2$$

Equating coefficients for different powers of x:

x^0: $2c - a - 3b = 2$

x^1: $6c + b = 0$

x^2: $-c = 1$

Solving simultaneously

$$c = -1,$$

$$b = -6c = 6$$

$$a = 2c - 3b - 2 = -2 - 18 - 2 = -22$$

and the general solution

$$y = Ae^{½(3+\sqrt{13})x} + Be^{½(3-\sqrt{13})x} - x^2 + 6x - 22.$$

Trigonometric f(x)

Consider $y'' + 2y' + y = \sin \omega t$

$\Rightarrow \quad \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0$

$\Rightarrow \quad \lambda = -1$, twice

$\Rightarrow$ C.F.: e^{-x} and xe^{-x}.

Trial solution $y = a \sin \omega t + b \cos \omega t$ and substitute

$$-\omega^2(a \sin \omega t + b \cos \omega t) + 2\omega(a \cos \omega t - b \sin \omega t) + a \sin \omega t + b \cos \omega t = \sin \omega t$$

Equating terms

sin ωt: $\quad -a\omega^2 - 2b\omega + a = (1-\omega^2)a - 2b\omega = 1$

cos ωt: $\quad -b\omega^2 + 2a\omega + b = 2a\omega + (1-\omega^2)b = 0$

Simultaneous solution gives

$$a = \frac{1-\omega^2}{\left(1+\omega^2\right)^2},\ b = \frac{-2\omega}{\left(1+\omega^2\right)^2}$$

and hence the particular integral

$$y = \frac{1-\omega^2}{\left(1+\omega^2\right)^2}\sin \omega t - \frac{2\omega}{\left(1+\omega^2\right)^2}\cos \omega t$$

Thus the general solution is

$$y = \left(A + Bt\right)e^{-t} + \frac{1-\omega^2}{\left(1+\omega^2\right)^2}\sin \omega t - \frac{2\omega}{\left(1+\omega^2\right)^2}\cos \omega t.$$

If the right-hand side contains a number of terms added together, for example $f(x) = x^3 + \cos x$, then the linearity of the equation means that the particular integral will be the sum of the particular integrals for the corresponding equations with $f(x) = x^3$ and $f(x) = \cos x$.

Multiple terms

Consider $$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x^4 + e^{3x} + \cos x.$$

Remember the equation is linear.

The first term in $f(x)$ suggests trying a polynomial of the form

$$a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

Substitution gives

$$(12a_4x^2 + 6a_3x + 2a_2) + (4a_4x^3 + 3a_3x^2 + 2a_2x + a_1) + (a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0) = x^4$$

x^4: $a_4 = 1$ $\Rightarrow a_4 = 1$

x^3: $4a_4 + a_3 = 4 + a_3 = 0$ $\Rightarrow a_3 = -4$

x^2: $12a_4 + 3a_3 + a_2 = 12 - 12 + a_2 = 0$ $\Rightarrow a_2 = 0$

x: $6a_3 + 2a_2 + a_1 = -24 + a_1 = 0$ $\Rightarrow a_1 = 24$

x^0: $2a_2 + a_1 + a_0 = 24 + a_0 = 0$ $\Rightarrow a_0 = -24$

The second term in $f(x)$ suggests a trial function of the form be^{3x}, substitution giving

$$9be^{3x} + 3be^{3x} + be^{3x} = 13be^{3x} = e^{3x} \qquad \Rightarrow b = 1/13.$$

The final term in $f(x)$ suggests a trial function of the form $c_1\cos x + c_2\sin x$. Substitution gives

$$(-c_1 \cos x - c_2 \sin x) + (-c_1 \sin x + c_2 \cos x) + (c_1\cos x + c_2\sin x) = \cos x$$

$\cos x$: $-c_1 + c_2 + c_1 = c_2 = 1$ $\Rightarrow c_2 = 1$

$\sin x$: $-c_2 - c_1 + c_2 = -c_1 = 0$ $\Rightarrow c_1 = 0$

Thus the full particular integral is

$$y = x^4 - 4x^3 + 24x - 24 + \tfrac{1}{13}e^{3x} + \sin x.$$

Of course, we need the complementary functions to complete the solution:

$$\lambda^2 + \lambda + 1 = 0$$

⇒ $$\lambda = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 1} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2},$$

etc. Had we been wise then we would have determined this first!

f(x) proportional to complementary function

Consider $$y'' + y = \cos t$$

Auxiliary equation $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

The complementary functions $\sin t$ and $\cos t$, so trial function cannot simply be $a \sin t + b \cos t$. Try instead $y = a\,t \sin t + b\,t \cos t$. Substituting

$$(2a \cos t - a\,t \sin t - 2b \sin t - b\,t \cos t) + (a\,t \sin t + b\,t \cos t) = \cos t$$

Equating terms

sin t: $$-at - 2b + at = -2b = 0$$

cos t: $$2a - b\,t + bt = 1$$

⇒ $$a = ½,\ b = 0$$

$$y = A \sin t + B \cos t + ½\,t \sin t.$$

Note: It is always worth determining the complementary functions *first* so as to avoid unnecessary work when working out the particular integral.

Repeated root and right-hand side of form of complementary function

Not only does the equation

$$y'' + 2y' + y = e^{-t}$$

have an auxiliary equation with a repeated root, $\lambda = -1$ and hence complementary functions are e^{-t} and te^{-t}, but the right-hand side $f(t)$ has the same form as the first complementary function.

We cannot simply try te^{-t} instead as this is the other complementary function. Instead, we can extend the idea we used previously and try $(a + bt + ct^2)e^{-t}$:

$$(a-2b+2c + (b-4c)t + ct^2)e^{-t} + 2(-a+b + (-b+2c)t - ct^2)e^{-t} + (a + bt + ct^2)e^{-t} = e^{-t}$$

Equating coefficients

$$a-2b+2c - 2a+2b + a = 2c = 1$$

$$b-4c - 2b+4c + b = 0 = 0$$

$$c - 2c + c = 0 = 0$$

Note that the last two do not tell us anything: this is no surprise, since we know that a and b are arbitrary as they represent the homogeneous solution.

$\Rightarrow$ $$c = ½$$

Thus the general solution$y = (A + Bt + ½\, t^2)\, e^{-t}$.

If $f(t)$ had been $e^{-t} + te^{-t}$, then we would have had to include a t^3e^{-t} as well as t^2e^{-t} in our trial function.

In general, if $f(x) = e^{sx}$ and $y = a\,e^{sx}$ does not work as a particular integral, try $y = ax\,e^{sx}$. If that does not work, try $y = ax^2\,e^{sx}$, *etc.*

The trial functions necessary to match polynomial terms in $f(x)$ are not affected by the presence of repeated roots. If the right-hand side contains a term in x^n, then the trial function should contain a polynomial of the form $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$.

1.3.6 Second order equations with variable coefficients

You will not be asked to solve a general second-order equation if the coefficients are not constant *unless* it can be written as

$$y'' = F(x,y').$$

Note that F does not contain explicit dependence on y. The tools we have already discussed can cope with this. We begin by writing $\zeta = y'$, from which we have a first order equation for ζ,

$$\zeta' = F(x,\zeta),$$

which we solve for ζ, then it is simply a matter of solving

$$\frac{dy}{dx} = \zeta(x)$$

to obtain

$$y = \int \zeta \, dx + c.$$

Note that there will be *two* constants of integration: one in ζ and one from integrating ζ.

Example

Solve
$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = x.$$

Set $\zeta = dy/dx$ so that

$$\frac{d\zeta}{dx} + \frac{1}{x}\zeta = x.$$

The integrating factor for this is

$$I(x) = \exp\left(\int^x p(\xi)\,d\xi\right) = \exp\left(\int^x \frac{1}{\xi}\,d\xi\right) = \exp(\ln x) = x$$

so

$$\frac{d}{dx}(I\zeta) = \frac{d}{dx}(x\zeta) = Ix = x^2$$

$\Rightarrow$ $$x\zeta = \tfrac{1}{3}x^3 + a$$

$\Rightarrow$ $$\zeta = \tfrac{1}{3}x^2 + \frac{a}{x}$$

Recalling that $dy/dx = \zeta$, so

$$\frac{dy}{dx} = \zeta = \tfrac{1}{3}x^2 + \frac{a}{x}$$

$\Rightarrow$ $$y = \tfrac{1}{9}x^3 + a\ln x + b$$

1.3.7 Second order strategy

For *constant coefficients*

1. Write equation in standard form: $y'' + py' + qy = f(x)$

2. Write down and solve auxiliary equation for λ.

3. If λ_1, λ_2 real and distinct, then C.F. $y = A\exp(\lambda_1 x) + B\exp(\lambda_2 x)$

4. If repeated root, then C.F. $y = (A + Bx)\exp(\lambda x)$

5. If complex roots $\lambda_1, \lambda_2 = a \pm ib$, then C.F.
$y = e^{ax}(A\cos(bx) + B\sin(bx))$

6. Look for forms of PI matching right-hand side. If right-hand side has terms proportional to C.F.s, then try multiplying by x, *etc.*

7. Impose boundary and/or initial conditions

When *coefficients are not constant* but can write $y'' = F(x,y')$.

1. Substitute $\zeta = y'$ so that we have a first order equation for ζ

2. Solve the equation for ζ using the standard techniques for a first order equation. This solution will contain one arbitrary constant.

3. Solve $y' = \zeta$. Remember that there will now be two constants of integration.

5. Impose boundary and/or initial conditions.

When a *form of solution* is suggested

1. Substitute the suggested form $y = s(x)$ into the equation.

2. Look for an algebraic relationship that needs to be satisfied.

3. Solve the algebraic equation and substitute this back into the suggested form.

4. Impose boundary and/or initial conditions.

1.3.8 Second order Tripos examples

These questions are often very easy, provided you can do the standard manipulations.

2001 Paper 2

Find the general solutions of

(a) $$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=x-e^{-x}$$

(b) $$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=x^2+2e^{-x}$$

Solution to (a)

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=x-e^{-x}$$

The auxiliary equation

$$\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$$

has solutions $\lambda_1 = -1$ and $\lambda_2 = -2$ giving the complementary functions

$$y = Ae^{-x} + Be^{-2x}.$$

The right-hand side contains two terms. To match the x term, we require the trial solution to contain both. Since e^{-x} is one of the complementary functions, try $y = ax + b + cxe^{-x}$:

$$\begin{aligned}\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y &= c\frac{d}{dx}\left(e^{-x}-xe^{-x}\right)\\ &\quad +3\left(a+ce^{-x}-cxe^{-x}\right)+2\left(ax+b+cxe^{-x}\right)\\ &= c\left(-e^{-x}-e^{-x}+xe^{-x}\right)\\ &\quad +3\left(a+ce^{-x}-cxe^{-x}\right)+2\left(ax+b+cxe^{-x}\right)\\ &= ce^{-x}+2ax+\left(3a+2b\right)\\ &= x-e^{-x}\end{aligned}$$

x: $2a = 1$ $\Rightarrow a = ½$

x^0: $3a + 2b = 0$ $\Rightarrow b = -3/4$

xe^{-x}: $c = -1$ $\Rightarrow c = -1.$

Hence the general solution is

$$y = Ae^{-x} + Be^{-2x} + \tfrac{1}{2}x - \tfrac{3}{4} - xe^{-x}.$$

Solution to (b)

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 + 2e^{-x}$$

The auxiliary equation is

$$\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0$$

so there is a repeated root $\lambda = -1$ and the complementary functions must be e^{-x} and xe^{-x}.

For the particular integral we must have polynomial terms up to x^2, while to match e^{-x} we must introduce x^2e^{-x} since both e^{-x} and xe^{-x} are complementary functions. Hence try $y = ax^2 + bx + c + dx^2e^{-x}$:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{d}{dx}\left(2ax + b + 2dxe^{-x} - dx^2e^{-x}\right)$$
$$+2\left(2ax + b + 2dxe^{-x} - dx^2e^{-x}\right) + \left(ax^2 + bx + c + dx^2e^{-x}\right)$$

$$= \left(2a + 2de^{-x} - 2dxe^{-x} - 2dxe^{-x} + dx^2e^{-x}\right)$$
$$+2\left(2ax + b + 2dxe^{-x} - dx^2e^{-x}\right)$$
$$+\left(ax^2 + bx + c + dx^2e^{-x}\right)$$
$$= ax^2 + (4a + b)x + (2a + 2b + c) + 2de^{-x}$$
$$= x^2 + 2e^{-x}$$

Matching coefficients

x^2: $a = 1$ $\Rightarrow a = 1$

x: $4a + b = 4 + b = 0$ $\Rightarrow b = -4$

x^0: $2a + 2b + c = 2 - 8 + c = 0$ $\Rightarrow c = 6$

x^2e^{-x}: $2d = 2$ $\Rightarrow d = 1$

Hence the general solution (sum of complementary functions and particular integral) is

$$y = (A + Bx)e^{-x} + x^2 - 4x + 6 + x^2e^{-x}.$$

2003 Paper 2

Consider the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2x\sin x.$$

Find a particular solution, of the form

$$y(x) = (a + bx)\sin x + (c + dx)\cos x$$

where the constants a, b, c and d are to be determined.

Hence find the solution $y(x)$ that satisfies the initial conditions $y = 0$ and $dy/dx = 0$ at $x = 0$.

Solution

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2x\sin x.$$

given $$y(x) = (a+bx)\sin x + (c+dx)\cos x$$

We already have the form of the solution, so simply have to substitute it into the original equation and equate the coefficients.

$$y(x) = a\sin x + c\cos x + bx\sin x + dx\cos x$$

$$\begin{aligned}\frac{dy}{dx} &= a\cos x + b\sin x + bx\cos x - c\sin x + d\cos x - dx\sin x \\ &= (b-c)\sin x + (a+d)\cos x - dx\sin x + bx\cos x\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= (b-c)\cos x - (a+d)\sin x - d\sin x - dx\cos x + b\cos x - bx\sin x \\ &= -(a+2d)\sin x + (2b-c)\cos x - bx\sin x - dx\cos x\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y &= \left[-(a+2d) - 2(b-c) + a\right]\sin x \\ &\quad + \left[(2b-c) - 2(a+d) + c\right]\cos x \\ &\quad + \left[-b + 2d + b\right]x\sin x \\ &\quad + \left[-d - 2b + d\right]x\cos x \\ &= 2x\sin x\end{aligned}$$

Equating coefficients

sin x: $-2b + 2c - 2d = 0$

cos x: $-2a + 2b - 2d = 0$

x sin x: $2d = 2$ $\Rightarrow d = 1$

x cos x: $-2b = 0$ $\Rightarrow b = 0$

Substituting back $a = b - d = -1$

$c = b + d = 1$

Hence, the particular integral is

$$y = -\sin x + (1 + x)\cos x.$$

To complete the solution, we need to determine the complementary functions of

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2x\sin x$$

The auxiliary equation gives $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$, which has a double root $\lambda = 1$. Hence the complementary functions are e^x and xe^x, so the general solution is

$$y = (A + Bx)e^x - \sin x + (1 + x)\cos x.$$

The initial conditions require $y = 0$ and $dy/dx = 0$ at $x = 0$. The first of these requires

$$0 = (A + 0) - 0 + (1 + 0) = A + 1 \qquad \Rightarrow A = -1.$$

For the second we require

$$\frac{dy}{dx} = Ae^x + Be^x + Bxe^x - \cos x - \sin x + \cos x - x\sin x$$
$$= (A + B)e^x + Bxe^x - (1 + x)\sin x$$

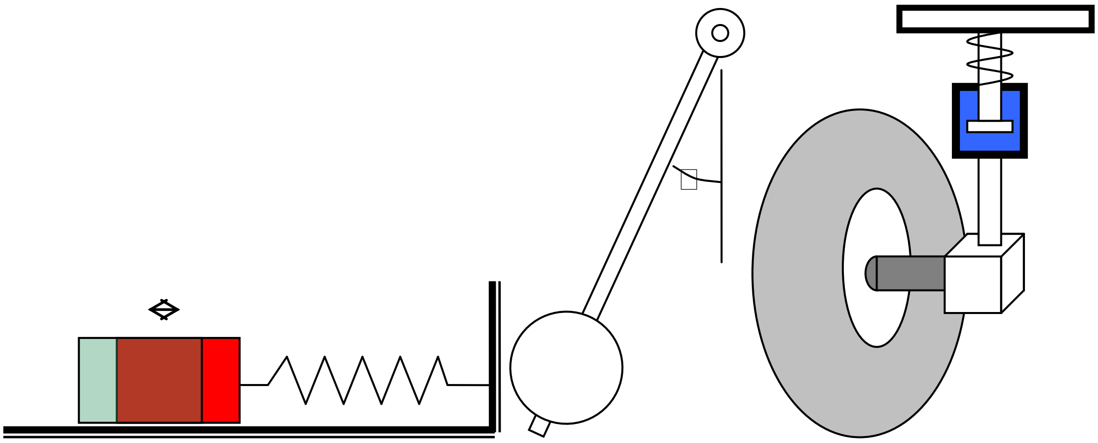
to vanish, so $\quad 0 = (A + B) + 0 - 0 = -1 + B \qquad \Rightarrow B = 1$

Hence the specific solution satisfying the boundary conditions is

$$y = (x - 1)e^x - \sin x + (1 + x)\cos x.$$

1.3.9 Unforced oscillators

As noted in §1.3.3, the behaviour of a mass on a spring may be modelled by a second-order linear ordinary differential equation.



If y is the location of the mass, then its acceleration is $a = d^2y/dt^2$, and we assume the force imparted by the spring is $-Ky$, where K is the spring constant (force per unit extension). We shall also assume that the mass is sitting on a lubricating film (*e.g.* oil) or attached to a dashpot so that the friction is proportional to the speed, giving drag force $D = 2J\,dy/dt$, say. Following Newton, $F = ma$, where F is the net force, m the mass and a the acceleration. Hence the natural response of the system (where there is no external forcing) is governed by

$$-Ky - 2J\frac{dy}{dt} = m\frac{d^2y}{dt^2}.$$

If $k^2 = K/m$ and $\mu = J/m$, then

$$\frac{d^2y}{dt^2} + 2\mu\frac{dy}{dt} + k^2y = 0.$$

We get an equation of the same form in a variety of situations, from clock pendulum to car suspension and washing machines.

The auxiliary equation

$$\lambda^2 + 2\mu\lambda + k^2 = 0 \qquad (**)$$

has solutions $$\lambda = -\mu \pm \sqrt{\mu^2 - k^2}\,.$$

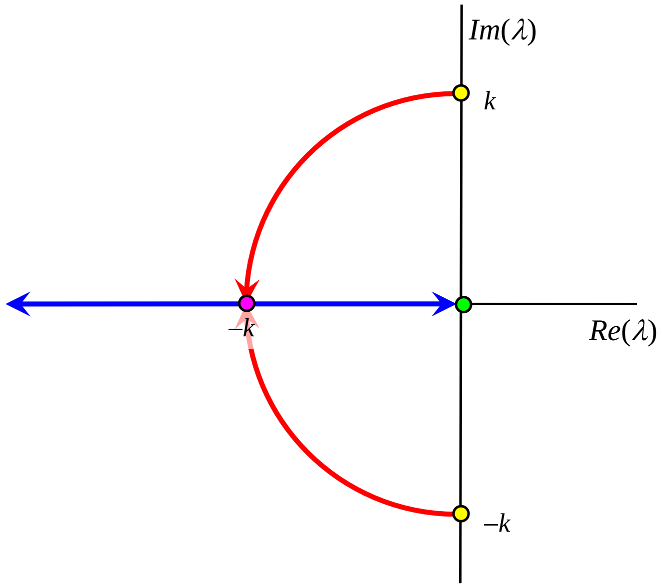


Figure 3: Changes in λ in the complex plane as μ is increased for constant k.

Undamped oscillation

When $\mu = 0$, then $\lambda = \pm ik$, and the homogeneous problem has oscillatory solutions

$$y = A \cos kt + B \sin kt$$

with frequency k rad/s (if time t is measured in seconds).

This form of motion is referred to as *simple harmonic motion* or *simple harmonic oscillation*.

Damped oscillation

When $\mu < k$, the auxiliary equation has complex solutions

$$\lambda = -\mu \pm i\sqrt{k^2 - \mu^2}\,,$$

and the homogeneous problem has oscillatory solutions that decay with time

$$y = e^{-\mu t}(A \cos \Omega t + B \sin \Omega t),$$

where $\Omega^2 = k^2 - \mu^2$. Note that the damping has reduced the frequency.

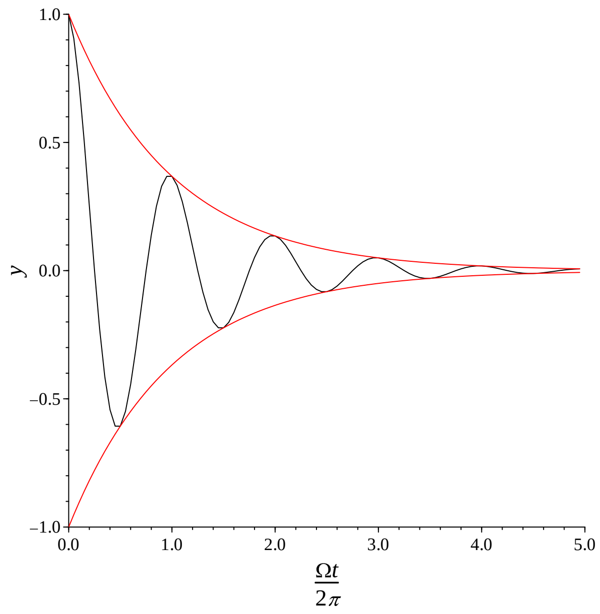


Figure 4: Damped oscillator.

Critical damping

The decay rate of the oscillation increases as μ increases and the frequency decreases until $\mu = k$, at which point the system is said to be *critically damped* (oscillations have zero frequency).

When $\mu = k$ the characteristic equation gives a double root $\lambda = \mu = k$ and the general solution is

$$y = (A + Bt)\, e^{-\mu t}.$$

Note: Car suspension is close to critically damped. Too little damping will cause the car to *bounce*, while too much damping will limit the suspension's ability to absorb shocks. Of course, since $k^2 = K/m$ and $\mu = J/m$, then the point at which critical damping is achieved depends on the mass of the car *and* load.

Over damping

When $\mu > k$ the system is *over damped* and there are no oscillations, only an exponential decay. The solutions to the auxiliary equation are purely real

$$\lambda = -\mu \pm \sqrt{\mu^2 - k^2}\,.$$

giving $y = e^{-\mu t}(A \sinh \gamma t + B \cosh \gamma t)$,

where $\gamma^2 = \mu^2 - k^2$.

When $\mu \gg k$, then

$$\lambda = -\mu\left[1 \pm \sqrt{1 - \left(\frac{k}{\mu}\right)^2}\right]$$
$$\approx -\mu\left[1 \pm \left(1 - \frac{1}{2}\left(\frac{k}{\mu}\right)^2\right)\right]$$

giving $\lambda_1 \approx -2\mu$ and $\lambda_2 \approx -\tfrac{1}{2} k^2/\mu$. The second of these gives a very slow decay rate.

In this limit we are close to solving

$$\ddot{y} + 2\mu\dot{y} = 0$$

$$\Rightarrow \quad \frac{1}{\dot{y}}\ddot{y} = -2\mu \quad \Rightarrow \quad \dot{y} = ae^{-2\mu t}$$

$$\Rightarrow \quad y = Ae^{-2\mu t} + B.$$

1.3.10 Forced oscillators

In the previous section we considered free oscillations, but of course something like a car or a washing machine has a continuous forcing. Suppose this forcing is sinusoidal in nature, then

$$\frac{d^2y}{dt^2} + 2\mu\frac{dy}{dt} + k^2 y = \sin \omega t\,.$$

This has the same complementary functions as before, plus a particular integral. Try $y = a \sin \omega t + b \cos \omega t$,

$$\Rightarrow -\omega^2(a \sin \omega t + b \cos \omega t) + 2\,\omega\mu(a \cos \omega t - b \sin \omega t)$$
$$+ k^2(a \sin \omega t + b \cos \omega t)$$
$$= \sin \omega t$$

$\sin \omega t$: $(k^2-\omega^2)a - 2\omega\mu b = 1$

$\cos \omega t$: $2\omega\mu a + (k^2-\omega^2)b = 0$

$$\Rightarrow y = \frac{k^2-\omega^2}{\left(k^2-\omega^2\right)^2+4\mu^2\omega^2}\sin\omega t - \frac{2\mu\omega}{\left(k^2-\omega^2\right)^2+4\mu^2\omega^2}\cos\omega t$$

Undamped solution

Noting that the particular integral is the *steady* (large time) solution, we can see that for $\mu = 0$ the solution tends towards the steady oscillation

$$y = \frac{1}{k^2-\omega^2}\sin\omega t,$$

which is singular if the forcing frequency ω matches the frequency of free oscillations, k. This situation is referred to as *resonance*.

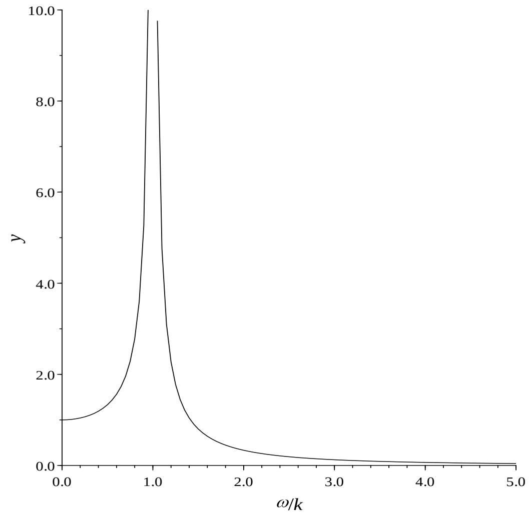


Figure 5: Undamped forced oscillation.

Note that as the forcing frequency comes closer and closer to the resonant frequency, the amplitude of the particular integral increases dramatically.

Damped oscillator

Damping prevents the amplitude from growing without bound as resonance is approached.

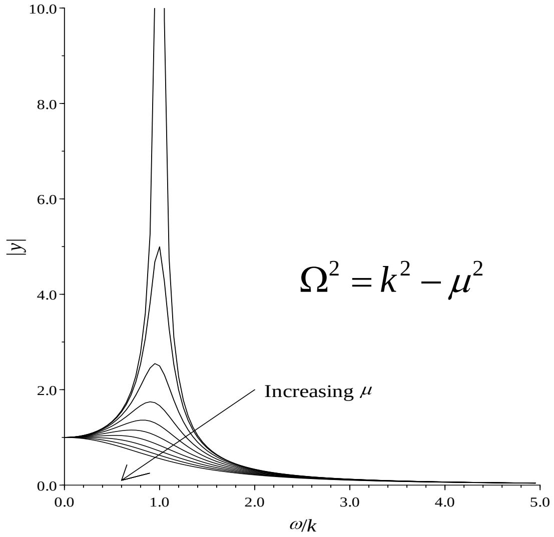


Figure 6: Damped forced oscillation showing effect of increasing the damping μ.

1.4 Systems of equations*

A system of two coupled first-order linear equations can be converted into a single second-order linear equation.

Consider $y(t)$ and $z(t)$ given by

$$\frac{dy}{dt} + ay + bz = 0,$$

$$\frac{dz}{dt} + py + qz = 0$$

where a, b, p and q are constants.

To determine the second-order equation for y, we begin by differentiating the first order equation for y:

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* Material in this section is border line on being examinable; questions have not been set on it in recent history.

$$\frac{d^2y}{dt^2}+a\frac{dy}{dt}+b\frac{dz}{dt}=0.$$

We can find a second expression for dz/dt by eliminating z between the original pair of equations:

$$q\left[\frac{dy}{dt}+ay+bz\right]-b\left[\frac{dz}{dt}+py+qz\right]=0,$$

$$\Rightarrow \qquad b\frac{dz}{dt}=q\frac{dy}{dt}+(aq-bp)y,$$

and use this in the second-order equation to get

$$\begin{aligned}\frac{d^2y}{dt^2}+a\frac{dy}{dt}+b\frac{dz}{dt}&=\frac{d^2y}{dt^2}+a\frac{dy}{dt}+\left[q\frac{dy}{dt}+(aq-bp)y\right]\\&=\frac{d^2y}{dt^2}+(a+q)\frac{dy}{dt}+(aq-bp)y\\&=0\end{aligned}$$

A similar equation can be determined for z:

$$\frac{d^2z}{dt^2}+(a+q)\frac{dz}{dt}+(aq-bp)z=0.$$

Note that the two equations are identical, except for z replacing y. This does *not* mean the two solutions are identical, but does mean the same form for the complementary solutions and the same time scales. That the timescales are the same is an inevitable consequence of the coupling between the equations.

The equations can be solved using the normal method for second-order equations, with appropriate initial conditions being used to determine the *two* constants of integration. Note that if we have $y(0)$ and $z(0)$ specified, then the first equation can be used to determine dy/dt at t=0, while the second to determine dz/dt at t = 0, so we know compatible initial conditions for y, dy/dt, z and dz/dt.

Aside – not examinable

There is an easier way of handling systems of equations such as this using the ideas associated with systems of linear algebraic equations. This approach, however, is beyond the present course and you will not be introduced to some of the prerequisite material (such as eigenvalues) until next term.

1.5 Higher order equations*

Higher order equations are beyond the scope of the NST1A course, but a few words on linear equations with constant coefficients can help the understanding of second-order equations.

Consider the homogeneous *n*th-order linear ordinary differential equation,

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-2}} + \cdots + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

where the coefficients a_0, a_1,…, a_{n-1} are all constant. As with the second-order equation, we try a solution of the form $y = e^{\lambda x}$ and recover the *n*th-order polynomial auxiliary equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0 = 0.$$

Solving this equation for λ will yield n roots, $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_n$, in the complex plane. These roots are not necessarily distinct. (Of course finding these roots may be difficult!) Each of these roots corresponds to a complementary function so that the general solution will be

$$y = A_1 e^{\lambda_1 x} + A_2 e^{\lambda_2 x} + \cdots + A_n e^{\lambda_n x},$$

if all the roots are distinct.

*This material could only be examined if you were told the form of solution to try.

Example

Find the general solution of the fourth-order equation

$$\frac{d^4y}{dt^4}+2\frac{d^2y}{dt^2}+y=0.$$

Trying the solution $y = e^{\lambda t}$ leads the to auxiliary equation

$$\lambda^4+2\lambda^2+1=\left(\lambda^2+1\right)^2=0.$$

Solution of this obviously requires $\lambda^2 = -1$, so the four roots are $\lambda = -i, -i, i, i$. Note that each of the two distinct roots are repeated. As with the second-order equations, the repeated roots introduce pairs of complementary functions of the form e^{it} and te^{it}. Expressing the general solution in trigonometric functions we have

$$y=(A+Bt)\cos t+(C+Dt)\sin t.$$

1.6 Other forms of linear equations*

There are other forms of linear equations that can be approached in a similar manner to that we have used here for the second order equations.

We know how to solve the first order homogeneous linear equation

$$\frac{dy}{dx}+a\frac{y}{x}=0$$

$$\Rightarrow \qquad \frac{1}{y}\frac{dy}{dx}=-\frac{a}{x}$$

$$\Rightarrow \qquad \ln|y|=-a\ln x+c$$

$$\Rightarrow \qquad y=Ax^{-a}.$$

* This material could only be examined if you were told the form of solution to try.

Now consider the second order homogeneous linear equation

$$\frac{d^2y}{dx^2}+\frac{4}{x}\frac{dy}{dx}+\frac{2}{x^2}y=0$$

and try a solution of the form $y = x^{\lambda}$:

Substituting

$$\lambda(\lambda-1)x^{\lambda-2}+\frac{4}{x}\lambda x^{\lambda-1}+\frac{2}{x^2}x^{\lambda}=\left(\lambda(\lambda-1)+4\lambda+2\right)x^{\lambda-2}=0$$

For a non trivial solution, we require the roots of the auxiliary equation

$$\lambda(\lambda-1)+4\lambda+2=\lambda^2+3\lambda+2=(\lambda+1)(\lambda+2)=0$$

giving $\lambda = -1, -2$. Thus the complementary functions are x^{-1} and x^{-2} and the general solution is

$$y=\frac{A}{x}+\frac{B}{x^2}$$

Special functions (beyond the scope of this course) have been developed to tackle a wide range of linear equations with non-constant coefficients.

2. Functions of several variables

2.1 Introduction

Problems, whether physical, biological or of some other origin, are frequently functions of more than one variable. *Steady* problems may be functions of two, three (or more!) spatial variables. *Transient* problems may be functions of time and one or more spatial variables. Even if space and time are not important, multiple variables and/or parameters may be important.

In this chapter we extend the ideas we have already encountered for functions of a single variable to include functions of more than one variable. Because it is often easier to visualise the problem, many of the cases we shall look at will be phrased in terms of space and time.

2.2 Differentiation

2.2.1 Partial derivatives

We need to use *partial derivatives* if we are differentiating functions of more than one independent variable.

Ordinary differentiation

Recall that for a function $f(x)$ of a single variable we define the derivative as

$$\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Partial differentiation

Frequently functions may depend on more than one variable. How do we differentiate these?

Consider $f(x,y)$. This might, for example, represent the height of a hill. In general, the *slope* will depend on the direction in which we are looking.

We define the *partial derivative* of *f* with respect to *x* as

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}.$$

This is the slope *in the x direction* at a given *x*,*y*. Put another way, we hold *y* constant and differentiate *f* as though it is a function only of *x*.

The quantity ∂*f*/∂*x* is referred to as "*the partial derivative of f with respect to x*". The curved ∂ itself can be referred to as "*partial* " or "*del* ", but often we will just call it "*d* ".

Similarly the partial derivative with respect to *y* is computed by holding *x* constant:

$$\frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}.$$

Note the use of ∂ in place of *d* to represent the derivative.

Other common notations: $\frac{\partial f}{\partial x}$ f_x $f_{,x}$ $\partial_x f$ and $\frac{\partial f}{\partial y}$ f_y $f_{,y}$ $\partial_y f$.

Example A: partial derivative as a limit

Consider $f(x,y) = 1 - x^2 - x\sin y + y^3$

Then

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{\left(1-(x+h)^2-(x+h)\sin y+y^3\right)-\left(1-x^2-x\sin y+y^3\right)}{h}$$

$$= -\lim_{h\to 0}\frac{(x+h)^2-x^2}{h} - (\sin y)\lim_{h\to 0}\frac{(x+h)-x}{h}$$

$$= -\lim_{h\to 0}(2x-h) - (\sin y)\lim_{h\to 0}(1)$$

$$= -2x - \sin y$$

which is the same as *df*/*dx* if we treat *y* as a constant.

Similarly,

$$\frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{\left(1-x^2 - x\sin(y+h) + (y+h)^3\right) - \left(1-x^2 - x\sin y + y^3\right)}{h}$$
$$= -x\lim_{h\to 0}\frac{\sin(y+h)-\sin y}{h} + \lim_{h\to 0}\frac{(y+h)^3 - y}{h}$$
$$= -x\cos y + 3y^2$$

Note that when computing the partial derivative we only differentiate with respect to an *explicit* dependence on a particular variable. For example, if we know not only that $f(x,y) = x^2 + y$, but also that we are interested only in values of y given by $y = 1-x$, then we ignore this secondary relationship between x and y when computing $\partial f/\partial x$ so that

$$\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial x}\right)_y = 2x\,.$$

The subscript y in the second expression above emphasises that we are holding y constant. We shall consider later (§2.2.3) how to calculate the rate of change of a function along a particular path that does not coincide with one of the function's independent variables.

Higher order partial derivatives

Higher order partial derivatives are defined in the natural way:

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad \text{(holding } y \text{ constant)}$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \qquad \text{(holding } x \text{ constant)}$$

$$\frac{\partial^2 f}{\partial x\partial y} = f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

These last two are *mixed partial derivatives*. To compute $\partial^2 f/\partial x \partial y$, first compute $\partial f/\partial y$ by holding x constant, then compute the $\partial/\partial x$ derivative of this by holding y constant.

Example B

Consider $f(x,y) = \exp(x^2 - y^2)$

then
$$\frac{\partial f}{\partial x} = 2x\exp\left(x^2 - y^2\right)$$
$$\frac{\partial f}{\partial y} = -2y\exp\left(x^2 - y^2\right)$$
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(2x\exp\left(x^2 - y^2\right)\right)$$
$$= 2\exp\left(x^2 - y^2\right) + 4x^2\exp\left(x^2 - y^2\right)$$
$$= \left(2 + 4x^2\right)\exp\left(x^2 - y^2\right)$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(-2y\exp\left(x^2 - y^2\right)\right)$$
$$= -2\exp\left(x^2 - y^2\right) + 4y^2\exp\left(x^2 - y^2\right)$$
$$= \left(-2 + 4x^2\right)\exp\left(x^2 - y^2\right)$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(-2y\exp\left(x^2 - y^2\right)\right)$$
$$= -4xy\exp\left(x^2 - y^2\right)$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(2x\exp\left(x^2 - y^2\right)\right)$$
$$= -4xy\exp\left(x^2 - y^2\right)$$

Note the last two are the same!

Example C

Consider
$$f(x,y)=\frac{1}{x+y^2}$$

So
$$f_x=\frac{-1}{\left(x+y^2\right)^2}$$

$$f_y=\frac{-2y}{\left(x+y^2\right)^2}$$

$$f_{xx}=\frac{\partial}{\partial x}\left(f_x\right)=\frac{\partial}{\partial x}\left(\frac{-1}{\left(x+y^2\right)^2}\right)=\frac{2}{\left(x+y^2\right)^3}$$

$$f_{yy}=\frac{\partial}{\partial y}\left(f_y\right)=\frac{\partial}{\partial y}\left(\frac{-2y}{\left(x+y^2\right)^2}\right)=\frac{-2}{\left(x+y^2\right)^2}+\frac{8y^2}{\left(x+y^2\right)^3}$$

$$f_{xy}=\frac{\partial}{\partial x}\left(f_y\right)=\frac{\partial}{\partial x}\left(\frac{-2y}{\left(x+y^2\right)^2}\right)=\frac{4y}{\left(x+y^2\right)^3}$$

$$f_{yx}=\frac{\partial}{\partial y}\left(f_x\right)=\frac{\partial}{\partial y}\left(\frac{-1}{\left(x+y^2\right)^2}\right)=\frac{4y}{\left(x+y^2\right)^2}$$

In the last two examples we see that $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$.

This property, the *symmetry of mixed partial derivatives* holds for **all** functions (provided certain smoothness properties are satisfied). Therefore, the following are all equivalent:

$$\frac{\partial^2 f}{\partial x \partial y}=\frac{\partial^2 f}{\partial y \partial x}=f_{xy}=f_{yx}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

Proof – You do not need to know this

So long as the limits are well behaved (the smoothness condition), we can use our fundamental definition to show that the order of differentiation does not matter:

$$\frac{\partial^2 f}{\partial x \partial y}=\lim_{h\to 0}\frac{\left.\frac{\partial f}{\partial y}\right|_{x+h,y}-\left.\frac{\partial f}{\partial y}\right|_{x,y}}{h}$$

$$=\lim_{h\to 0}\frac{\lim_{a\to 0}\frac{f(x+h,y+a)-f(x+h,y)}{a}-\lim_{a\to 0}\frac{f(x,y+a)-f(x,y)}{a}}{h}$$

$$=\lim_{h\to 0}\left(\lim_{a\to 0}\left[\frac{f(x+h,y+a)-f(x+h,y)-f(x,y+a)+f(x,y)}{ah}\right]\right)$$

$$=\lim_{a\to 0}\left(\frac{1}{a}\lim_{h\to 0}\left[\frac{f(x+h,y+a)-f(x,y+a)}{h}-\frac{f(x+h,y)-f(x,y)}{h}\right]\right)$$

$$=\lim_{a\to 0}\frac{\left.\frac{\partial f}{\partial x}\right|_{x,y+a}-\left.\frac{\partial f}{\partial x}\right|_{x,y}}{a}$$

$$=\frac{\partial^2 f}{\partial y \partial x}$$

Higher derivatives

These ideas for second-order derivatives extend naturally to higher derivatives, with, for example,

$$\frac{\partial^4 f}{\partial x^4} = f_{xxxx} = \frac{\partial}{\partial x}\left(f_{xxx}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(f_{xx}\right)\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(f_x\right)\right)\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right)\right).$$

Note that sometimes (as above) we might mix our notation, depending on what is most convenient, clear and compact.

Similarly, the order of differentiation does not matter so that

$$f_{xxyy} = f_{yyxx} = f_{xyxy} = f_{yxyx} = f_{xyyx} = f_{yxxy}.$$

More dimensions

There is nothing special about two-dimensions and these ideas readily extend naturally to functions of more than two variables. For example, the function $f(p,q,r,s)$ has four distinct first-order partial derivatives,

$$f_p = \frac{\partial f}{\partial p},\ f_q = \frac{\partial f}{\partial q},\ f_r = \frac{\partial f}{\partial r},\ f_s = \frac{\partial f}{\partial s},$$

and ten distinct second-order partial derivatives:

$$f_{pp} = \frac{\partial^2 f}{\partial p^2},\ f_{qq} = \frac{\partial^2 f}{\partial q^2},\ f_{rr} = \frac{\partial^2 f}{\partial r^2},\ f_{ss} = \frac{\partial^2 f}{\partial s^2},$$

$$f_{pq} = \frac{\partial^2 f}{\partial p \partial q} = \frac{\partial^2 f}{\partial q \partial p} = f_{qp}$$

$$f_{pr} = \frac{\partial^2 f}{\partial p \partial r} = \frac{\partial^2 f}{\partial r \partial p} = f_{rp}$$

$$f_{ps} = \frac{\partial^2 f}{\partial p \partial s} = \frac{\partial^2 f}{\partial s \partial p} = f_{sp}$$

$$f_{qr} = \frac{\partial^2 f}{\partial q \partial r} = \frac{\partial^2 f}{\partial r \partial q} = f_{rq}$$

$$f_{qs} = \frac{\partial^2 f}{\partial q \partial s} = \frac{\partial^2 f}{\partial s \partial q} = f_{sq}$$

$$f_{rs} = \frac{\partial^2 f}{\partial r \partial s} = \frac{\partial^2 f}{\partial s \partial r} = f_{sr}$$

Gradient vector

Later in the course (in Chapter 4) we will discuss the idea of the *gradient* of a function. This is essentially a vector constructed from the first order partial derivatives of the function. In particular, the gradient vector of $f(x,y)$ is

$$\operatorname{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right),$$

while for $g(p,q,r,s)$ it will be

$$\operatorname{grad} g = \nabla g = \left(\frac{\partial g}{\partial p}, \frac{\partial g}{\partial q}, \frac{\partial g}{\partial r}, \frac{\partial g}{\partial s} \right).$$

Hence, we can consider ∇ as a *vector of differentiation operators*. The symbol ∇ is known as *nabla*, but will frequently be referred to as *del*. With three independent variables (x,y,z), it represents

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right),$$

which, when acting on a scalar, will produce a vector result. In this context the operator is generally referred to as *grad*.

Later (Chapter 4) we will use the same symbol in a number of different contexts and a number of names, depending on its context.

This idea of the *gradient* is most often used when considering independent spatial variables, but can also be used for independent variables representing other quantities.

More notation

To emphasise what is being held constant we will sometimes write

$$\left(\frac{\partial f}{\partial x}\right)_y \quad \text{or} \quad \left.\frac{\partial f}{\partial x}\right|_y$$

in place of simply $\partial f/\partial x$ or f_x as a reminder that f is a function of both x and y.

> This notation is particularly useful when changing independent variables, *e.g.* for a function $f(x,y)$ if we change from (x,y) to (x,z), where z can be expressed as a function of x and y, *e.g.* $z = Z(x,y)$. In such cases, the distinction between $\left(\frac{\partial f}{\partial x}\right)_y$ and $\left(\frac{\partial f}{\partial x}\right)_z$ is very important. Holding z constant implies that moving in x also implies a movement in y. We shall return to this later in this chapter.

2.2.2 Differentials

We begin by considering the origin of a differential expression. While it is not necessary for you to be able to reproduce this background material, having seen it may help you understand and manipulate differential expressions.

Functions of a single variable

As you have seen previously, for a function of one variable, we can write a Taylor expansion (or Taylor series)

$$f(x_0 + h) = f(x_0) + h\,f'(x_0) + \tfrac{1}{2}h^2 f''(x_0) + \ldots$$

that gives a local approximation of f near x_0 as a polynomial expansion in h. [The number of terms we can potentially include in the expansion may be limited by the differentiability properties of f: to include a term of order h^n we require that the nth derivative, and all lower derivatives, exists.]

The *leading order variation* in f, *i.e.* the variation which dominates when h is sufficiently small, is the linear term so that

$$f(x_0 + h) - f(x_0) \approx hf'(x_0).$$

[We have used this already when defining the derivative f'.]

This linear approximation to f improves as h gets smaller in the sense that the *error term* divided by h also gets smaller. In particular, that

$$\frac{f(x_0+h)-\left[f(x_0)+hf'(x_0)\right]}{h} \to 0 \text{ as } h \to 0.$$

We write the approximation $f(x_0 + h) - f(x_0) \approx hf'(x_0)$ in shorthand as the *differential*

$$df = f'\,dx.$$

This differential can be interpreted as the change in the function f is equal to the derivative of f (*i.e.*, df/dx) multiplied by the change in x, a statement that is valid provided the function is differentiable and the change in x is small.

More specifically,

$$\begin{aligned} df &= \lim_{h\to 0} f(x+h) - f(x) \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} h \\ &= f'\,dx \end{aligned}$$

Aside

If we were to take a differential and integrate it,

$$\int_{f(x_0)}^{f(x)} df = \int_{x0}^{x} \frac{df}{dx} dx$$

then we just end up with

$$f(x) - f(x_0) = f(x) - f(x_0).$$

Functions of two variables

Suppose now we have a function of two variables, $f(x,y)$. By analogy with the linear Taylor series expansion for a function of a single variable, we expect the leading order behaviour of the function in the neighbourhood of the point (x_0,y_0) to be of the form

$$f(x_0 + h, y_0 + k) \approx f(x_0,y_0) + \alpha h + \beta k$$

where α and β are suitable constants.

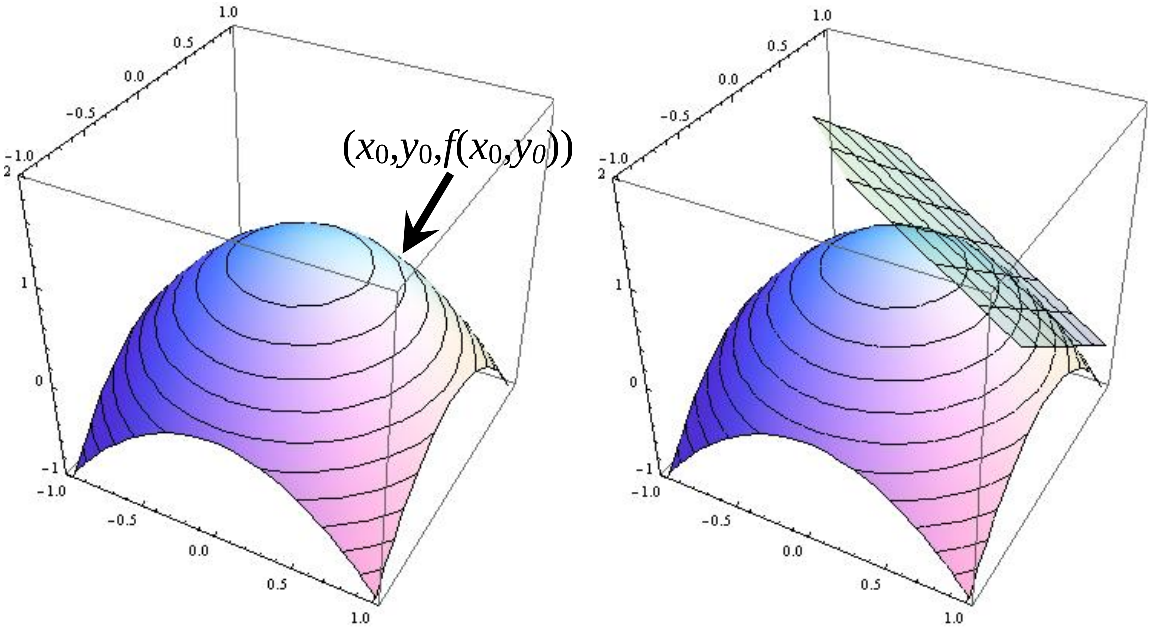


Figure 7: Linear approximation to a surface as a tangent plane.

This approximation represents a *tangent plane* that touches the function $f(x,y)$ at (x_0,y_0) and is tangential to it at that point.

If we denote the error in this approximation as

$$\delta(h,k) = f(x_0+h,y_0+k) - (f(x_0,y_0) + \alpha h + \beta k),$$

then the function $f(x,y)$ is said to be *differentiable* at (x_0,y_0) if the error decreases fast enough as h and k decrease. In particular, we require

$$\frac{\delta(h,k)}{\left(h^2+k^2\right)^{1/2}} \to 0 \text{ as } (h^2+k^2)^{1/2} \to 0.$$

To calculate α, we take $k = 0$, then

$$\frac{\delta(h,0)}{h} = \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} - \alpha \to 0 \text{ as } h \to 0.$$

Hence $$\alpha = \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} = \frac{\partial f}{\partial x}(x_0,y_0).$$

Similarly $$\beta = \lim_{k\to 0}\frac{f(x_0,y_0+k)-f(x_0,y_0)}{k} = \frac{\partial f}{\partial y}(x_0,y_0)$$

$$\Rightarrow\ f(x_0+h,y_0+k) \approx f(x_0,y_0) + h\frac{\partial f}{\partial x}(x_0,y_0) + k\frac{\partial f}{\partial y}(x_0,y_0).$$

Differential expressions – what you need to know

Using a similar shorthand as before, we can rewrite this as the *differential* expression

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

One literal interpretation of the above statement is that 0 = 0 (since we are dealing with the limits as things go to zero), but its value is as a statement about how the limit is approached. For example, if x and y are functions of some parameter t, it implies

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Clausius-Clapeyron relation

The vapour pressure above a liquid at temperature T is given by

$$P = P_0 \exp\left(\frac{H}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right),$$

where P_0 is the vapour pressure at temperature T_0, H is the latent heat (enthalpy) of vaporisation, and R the ideal gas constant.

Suppose P_0, T_0 and R are known exactly, but we only know T to within a 5% accuracy, and H to within a 1% accuracy. What then, is the uncertainty in the vapour pressure P?

We can therefore regard P as a function of T and H, and determine how this changes for small changes in these parameters. With $P = P(T,H)$¸ then

$$dP = \frac{\partial P}{\partial T} dT + \frac{\partial P}{\partial H} dH .$$

Now $$\frac{\partial P}{\partial T} = \frac{H}{RT^2} P_0 \exp\left(\frac{H}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right) = \frac{H}{RT^2} \times P$$

$$\frac{\partial P}{\partial H} = \frac{1}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right) P_0 \exp\left(\frac{H}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right) = \frac{1}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right) \times P$$

so $$dP = \left(\frac{H}{RT^2} dT + \frac{1}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right) dH\right) P$$

$$\Rightarrow \quad \frac{dP}{P} = \frac{H}{RT}\frac{dT}{T} + \frac{H}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\frac{dH}{H}$$

and the percentage uncertainty is

$$\frac{dP}{P} \approx \frac{H}{RT} \times 5\% + \frac{H}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right) \times 1\%$$

2.2.3 The chain rule

Functions of a single variable

Amongst the things we reviewed in §1.1.3 was the *chain rule* for a function of a single variable.

For $$z = g(h(x))$$

we found $$\frac{dz}{dx} = \frac{dg}{dh}\frac{dh}{dx}.$$

If we were to write this as a differential, it would be

$$dz = \frac{dg}{dh}\frac{dh}{dx}dx.$$

Here we shall extend the idea to functions of more than one variable.

Changing to polar coordinates

What happens if we wish to move in another direction, say at an angle θ to the x axis? Let $\mathbf{s} = s(\cos\theta, \sin\theta)$ be a vector in this direction, then the slope will be

$$\begin{aligned}
\frac{df}{ds} &= \lim_{h\to0}\frac{f(x+h\cos\theta, y+h\sin\theta)-f(x,y)}{h} \\
&= \lim_{h\to0}\frac{f(x+h\cos\theta, y+h\sin\theta)-f(x,y+h\sin\theta)+f(x,y+h\sin\theta)-f(x,y)}{h} \\
&= \lim_{h\to0}\frac{f(x+h\cos\theta, y+h\sin\theta)-f(x,y+h\sin\theta)}{h\cos\theta}\cos\theta+\lim_{h\to0}\frac{f(x,y+h\sin\theta)-f(x,y)}{h\sin\theta}\sin\theta \\
&= \lim_{h\cos\theta\to0}\frac{f(x+h\cos\theta,y)-f(x,y)}{h\cos\theta}\cos\theta+\lim_{h\sin\theta\to0}\frac{f(x,y+h\sin\theta)-f(x,y)}{h\sin\theta}\sin\theta \\
&= \frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta
\end{aligned}$$

Now since

$$x = s\cos\theta \Rightarrow dx/ds = \cos\theta$$

$$y = s\sin\theta \Rightarrow dy/ds = \sin\theta,$$

we may rewrite this as

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

We can get to this result in a much more straightforward way be employing the *chain rule*.

General change in variables

Suppose f is a function of x and y, and, in turn, x and y are functions of u and v, so we can write

$$f\left(x(u,v), y(u,v)\right) = \hat{f}(u,v).$$

Note that f and $\hat{f}$ represent the same function, just expressed differently.

We wish to calculate $\frac{\partial \hat{f}}{\partial u}$ and $\frac{\partial \hat{f}}{\partial v}$, although we will often (usually) just write these as $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$.

The changes in f due to the changes in u are due to both changes in x and changes in y. Similarly, the changes in f due to changes in v are due to both changes in x and in y. We can write these changes in x and y as

$$dx = \left(\frac{\partial x}{\partial u}\right)_v du + \left(\frac{\partial x}{\partial v}\right)_u dv$$

and

$$dy = \left(\frac{\partial y}{\partial u}\right)_v du + \left(\frac{\partial y}{\partial v}\right)_u dv,$$

while the change in f due to changes in x and y is

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy.$$

Substituting

$$df = \left(\frac{\partial f}{\partial x}\right)_y \left[\left(\frac{\partial x}{\partial u}\right)_v du + \left(\frac{\partial x}{\partial v}\right)_u dv\right] + \left(\frac{\partial f}{\partial y}\right)_x \left[\left(\frac{\partial y}{\partial u}\right)_v du + \left(\frac{\partial y}{\partial v}\right)_u dv\right]$$

and rearranging to collect together terms in *du* and *dv*:

$$\begin{aligned} df = &\left[\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial u}\right)_v\right] du \\ &+ \left[\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial v}\right)_u + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u\right] dv \end{aligned}.$$

The rate of change of *f* with *u* keeping *v* fixed is therefore

$$\left(\frac{\partial f}{\partial u}\right)_v = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial u}\right)_v.$$

Similarly $$\left(\frac{\partial f}{\partial v}\right)_u = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial v}\right)_u + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u.$$

These are expressions of the *chain rule*

A special case is when *x* and *y* are both functions of a single variable, *t*, say. Then $dx = (dx/dt)\,dt$ and $dy = (dy/dt)\,dt$, so

$$df = \left[\left(\frac{\partial f}{\partial x}\right)_y \frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_x \frac{dy}{dt}\right] dt.$$

[Note the use of dx/dt rather than $\partial x/\partial t$ as the former reflects that *x* is a function only of *t*.]

It is therefore obvious that the derivative

$$\frac{df}{dt} = \frac{d}{dt}\big(f\left(x(t), y(t)\right)\big) = \left(\frac{\partial f}{\partial x}\right)_y \frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_x \frac{dy}{dt}.$$

Again, we have used the *complete derivative* notation df/dt rather than the partial derivative notation $\partial f/\partial t$ since variations in t are responsible for *all* the variations in f.

Cartesian to polar coordinates

Change from Cartesian coordinates (x,y) to two-dimensional polar coordinates (r,θ).

We begin by writing

$$x = r\cos\theta, \quad y = r\sin\theta.$$

If the function f is specified in terms of x and y then, by the chain rule,

$$\left(\frac{\partial f}{\partial r}\right)_\theta = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial r}\right)_\theta + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial r}\right)_\theta$$

$$= \left(\frac{\partial f}{\partial x}\right)_y (\cos\theta) + \left(\frac{\partial f}{\partial y}\right)_x (\sin\theta)$$

and

$$\left(\frac{\partial f}{\partial \theta}\right)_r = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial \theta}\right)_r + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial \theta}\right)_r$$

$$= \left(\frac{\partial f}{\partial x}\right)_y (-r\sin\theta) + \left(\frac{\partial f}{\partial y}\right)_x (r\cos\theta)$$

Note that here we used the *old* variables (x,y) in terms of the *new* variables (r,θ).

Polar to Cartesian

If we are given the polar function $g(r,\theta)$ and wish to determine the Cartesian derivatives $(\partial g/\partial x)_y$ and $(\partial g/\partial y)_x$.

$$\left(\frac{\partial g}{\partial x}\right)_y = \left(\frac{\partial g}{\partial r}\right)_\theta \left(\frac{\partial r}{\partial x}\right)_y + \left(\frac{\partial g}{\partial \theta}\right)_r \left(\frac{\partial \theta}{\partial x}\right)_y$$

so we need $r = (x^2 + y^2)^{1/2}$ and $\theta = \tan^{-1}(y/x)$ and

$$\left(\frac{\partial g}{\partial x}\right)_y = \left(\frac{\partial g}{\partial r}\right)_\theta \frac{\partial}{\partial x}\left(\left(x^2+y^2\right)^{1/2}\right) + \left(\frac{\partial g}{\partial \theta}\right)_r \frac{\partial}{\partial x}\left(\tan^{-1}\frac{y}{x}\right)$$

$$= \left(\frac{\partial g}{\partial r}\right)_\theta \frac{x}{\left(x^2+y^2\right)^{1/2}} + \left(\frac{\partial g}{\partial \theta}\right)_r \frac{-y}{x^2+y^2}$$

$$= \left(\frac{\partial g}{\partial r}\right)_\theta \cos\theta + \left(\frac{\partial g}{\partial \theta}\right)_r \frac{-\sin\theta}{r}$$

Similarly,

$$\left(\frac{\partial g}{\partial y}\right)_x = \left(\frac{\partial g}{\partial r}\right)_\theta \left(\frac{\partial r}{\partial y}\right)_x + \left(\frac{\partial g}{\partial \theta}\right)_r \left(\frac{\partial \theta}{\partial y}\right)_x$$

$$= \left(\frac{\partial g}{\partial r}\right)_\theta \frac{\partial}{\partial y}\left(\left(x^2+y^2\right)^{1/2}\right) + \left(\frac{\partial g}{\partial \theta}\right)_r \frac{\partial}{\partial y}\left(\tan^{-1}\frac{y}{x}\right)$$

$$= \left(\frac{\partial g}{\partial r}\right)_\theta \frac{y}{\left(x^2+y^2\right)^{1/2}} + \left(\frac{\partial g}{\partial \theta}\right)_r \frac{x}{x^2+y^2}$$

$$= \left(\frac{\partial g}{\partial r}\right)_\theta \sin\theta + \left(\frac{\partial g}{\partial \theta}\right)_r \frac{\cos\theta}{r}$$

Contours

Given $z = h(x,y)$, can we find $(\partial x/\partial y)_z$ (*i.e.*, find the curve along which $z = const.$)?

Now
$$dz = \left(\frac{\partial h}{\partial x}\right)_y dx + \left(\frac{\partial h}{\partial y}\right)_x dy$$

so at constant z ($dz = 0$)

$$\begin{aligned}
0 &= \left(\frac{\partial h}{\partial x}\right)_y dx + \left(\frac{\partial h}{\partial y}\right)_x dy \\
&= \left(\frac{\partial h}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial h}{\partial y}\right)_x \left(\frac{\partial y}{\partial y}\right)_z dy \\
&= \left[\left(\frac{\partial h}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z + \left(\frac{\partial h}{\partial y}\right)_x\right] dy
\end{aligned}$$

$$\Rightarrow \qquad \left(\frac{\partial x}{\partial y}\right)_z = -\left(\frac{\partial h}{\partial y}\right)_x \Big/ \left(\frac{\partial h}{\partial x}\right)_y$$

Although the right hand side looks like we are simply dividing one fraction by another to eliminate *f*, such a simplistic interpretation can be misleading – note the minus sign!

Implicit and explicit dependence

We need to exercise a little more care when there is both implicit and explicit dependence on a parameter. Consider

$$f(x,y,t) = x^2 + y^2 + \sin t.$$

The differential in this case will be

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial t}dt$$

Suppose we are interested in df/dt in the case when x and y depend parametrically on t as $x = t$ and $y = 1 - t$.

We could, of course, simply substitute for x and y into the function f, then differentiate:

$$f = x^2 + y^2 + \sin t = (t)^2 + (1-t)^2 + \sin t$$

$$= 2t^2 - 2t + 1 + \sin t$$

$$\Rightarrow \qquad \frac{df}{dt} = 4t - 2 + \cos t.$$

Alternatively we can invoke to chain rule to write

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial t}dt \\ &= \frac{\partial f}{\partial x}\frac{dx}{dt}dt + \frac{\partial f}{\partial y}\frac{dy}{dt}dt + \frac{\partial f}{\partial t}dt \\ &= \left[\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}\right]dt \end{aligned}$$

so the *total derivative* with respect to time, given that $x = x(t)$ and $y = y(t)$, is

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{\partial y}{dt} + \frac{\partial f}{\partial t} \\ &= \left(\frac{\partial f}{\partial x}\right)_{y,t}\frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_{x,t}\frac{dy}{dt} + \left(\frac{\partial f}{\partial t}\right)_{x,y} \end{aligned}$$

The notation used on the second line serves to reinforce what is held constant during the formation of the partial derivatives. Substituting

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t} \\ &= (2x)_{x=t}(1) + (2y)_{y=1-t}(-1) + \cos t \\ &= 2t - 2(1-t) + \cos t \\ &= 4t - 2 + \cos t\end{aligned}$$

Crossing a hill

Consider a hill of height $h(x,y) = 1/(x^2 + 2y^4 + 1)$. Suppose our position as a function of time $t > 0$ is given by $x = (1 + t)^{½}$ and $y = 1 - t$.

What is our rate of change of height as a function of time, *i.e.*, what is dh/dt?

Using the chain rule

$$\begin{aligned}\frac{dh}{dt} &= \left(\frac{\partial h}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial h}{\partial y}\right)\frac{dy}{dt} \\ &= \left(\frac{-2x}{(x^2+2y^4+1)^2}\right)\left(\frac{1}{2(1+t)^{1/2}}\right) + \left(\frac{-8y^3}{(x^2+2y^4+1)^2}\right)(-1) \\ &= \frac{1}{(x^2+2y^4+1)^2}\left(\frac{-x}{(1+t)^{1/2}} + 8y^3\right)\end{aligned}$$

and substitute in for x and y

$$\frac{dh}{dt} = \frac{1}{\left((1+t)+2(1-t)^4+1\right)^2}\left(-\frac{(1+t)^{1/2}}{(1+t)^{1/2}}+8(1-t)^3\right)$$

$$= \frac{-1+8(1-t)^3}{\left(2+t+2(1-t)^4\right)^2}$$

$$= \frac{-(2t-1)\left(4t^2-10t+7\right)}{\left(2+t+2(1-t)^4\right)^2}$$

Note that this vanishes at $t = ½$.

Exercise: Check that you get the same result by substituting $x(t)$, $y(t)$ into $h(x,y)$ and differentiating the resulting function.

2.2.4 Reciprocity

Ordinary derivatives

Suppose $y = f(x)$, then the 'slope' is just $dy/dx = f'(x)$.

If instead, we were to write $x = f^{-1}(y) = g(y)$, then the 'inverse slope' would be $dx/dy = g'(y)$.

However, since $y = f(x) = f(g(y)) = y$, then by the chain rule

$$\frac{dy}{dy} = \frac{d}{dy}\left(f\left(g\left(y\right)\right)\right) = \frac{df\left(g\right)}{dg}\frac{dg\left(y\right)}{dy} = f'g' = \frac{df}{dx}\frac{dg}{dy} = \frac{dy}{dx}\frac{dx}{dy},$$

but as $dy/dy = 1$, then we recover the obvious statement that

$$\frac{dy}{dx}\frac{dx}{dy} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{dx/dy}$$

This is the 'reciprocity relation' for an ordinary derivative.

Partial derivatives

If z is a function of two variables, x and y, so that $z = z(x,y)$, then we can also treat x as a function of y and z (*i.e.*, $x = x(y,z)$) and y as a function of x and z (*i.e.*, $y = y(x,z)$). The corresponding differential expressions are

$$dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz\,,$$

$$dy = \left(\frac{\partial y}{\partial x}\right)_z dx + \left(\frac{\partial y}{\partial z}\right)_x dz\,,$$

$$dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy\,.$$

Now provided $(\partial y/\partial x)_z \neq 0$ (which is related to the condition of a monotonic relationship between y and x), we can rearrange the expression for dy to give

$$dx = \left[1\Big/\left(\frac{\partial y}{\partial x}\right)_z\right]dy - \left[\left(\frac{\partial y}{\partial z}\right)_x \Big/ \left(\frac{\partial y}{\partial x}\right)_z\right]dz\,.$$

Comparing with the original expression for dx shows

$$\left(\frac{\partial x}{\partial y}\right)_z = 1\Big/\left(\frac{\partial y}{\partial x}\right)_z\,,$$

$$\left(\frac{\partial x}{\partial z}\right)_y = -\left(\frac{\partial y}{\partial z}\right)_x \Big/ \left(\frac{\partial y}{\partial x}\right)_z\,.$$

The first of these is the *reciprocity relation*. It is effectively a generalisation of $dy/dx = 1/(dx/dy)$ for ordinary derivatives.

The second of these is the *cyclic relation*. Using the reciprocity relation, this can also be written as

$$\left(\frac{\partial x}{\partial z}\right)_y\left(\frac{\partial y}{\partial x}\right)_z\left(\frac{\partial z}{\partial y}\right)_x = -1 = \left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y .$$

Contours

In an earlier example we found the orientation $(\partial y/\partial x)_z$ of contours of $z = h(x,y)$. We could instead tackle this using the cyclic relation,

$$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = -1,$$

and rearranging it using the reciprocity relation to obtain

$$\left(\frac{\partial x}{\partial y}\right)_z = -1\Bigg/\left[\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y\right] = -\left(\frac{\partial z}{\partial y}\right)_x\Bigg/\left(\frac{\partial z}{\partial x}\right)_y$$

$$= -\left(\frac{\partial h}{\partial y}\right)_x\Bigg/\left(\frac{\partial h}{\partial x}\right)_y$$

Note that we cannot simply 'cancel' the h between the partial derivatives $\partial h/\partial y$ and $\partial h/\partial x$.

2.2.5 Exact differentials

A general differential expression of two variables might be written as

$$P(x,y)\, dx + Q(x,y)\, dy.$$

Is this differential expression *exact*? Here the term 'exact' means is there a function $f(x,y)$ such that

$$df = P(x,y)\, dx + Q(x,y)\, dy?$$

If the answer is 'yes', then from our earlier discussion in which we saw that

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy,$$

and noting that dx and dy can be regarded as independent quantities, then

$$\left(\frac{\partial f}{\partial x}\right)_y = P(x,y) \quad \text{and} \quad \left(\frac{\partial f}{\partial y}\right)_x = Q(x,y).$$

Recalling that the order of differentiation does not matter, *i.e.*

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

This is a *necessary condition* for an exact differential. If this condition is not satisfied, then the differential cannot be exact.

The relation $\partial P/\partial y = \partial Q/\partial x$ is also (under certain conditions) a *sufficient condition* for the differential to be exact. [Proving this requires defining a possible function f as an integral and then showing that the definition is unique; this proof is beyond the scope of this course.]

The *general condition for an exact differential* is that

$$P(x,y)\,dx + Q(x,y)\,dy \text{ is an exact differential}$$

$$\text{if and only if } \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0.$$

Example A

Consider $y\,dx - x\,dy$.

Is this an *exact differential* in that there is a function f such that $df = y\,dx - x\,dy$?

In the standard form, this gives $P(x,y) = y$ and $Q(x,y) = -x$, so $\partial P/\partial y = 1$ and $\partial Q/\partial x = -1$.

Since $\partial P/\partial y \neq \partial Q/\partial x$ then the differential is **not** exact.

Example B

Consider $$y\,dx + x\,dy.$$

Noting that $\dfrac{\partial P}{\partial y} = 1$ and $\dfrac{\partial Q}{\partial x} = 1$, then the differential is exact.

Can we find the corresponding function $f(x,y)$?

We have $$\frac{\partial f}{\partial x} = P(x,y) = y \quad \Rightarrow \quad f = xy + g(y)$$

where $g(y)$ is an arbitrary function of y, but a constant with respect to x.

Similarly, we have $$\frac{\partial f}{\partial y} = Q(x,y) = x \quad \Rightarrow \quad f = xy + h(x)$$

where $h(x)$ is an arbitrary function of x (constant with respect to y).

Since these two expressions for f must be equivalent (*i.e.*, $f = xy + g(y) = xy + h(x)$), then $g(y) = h(x) = const$ and

$$f = xy + c.$$

Use in solving differential equations

We can use the idea of exact differentials to construct the solution of a class of differential equations.

The differential $P(x,y)\,dx + Q(x,y)\,dy = 0$ is one way of writing the *ordinary differential equation*

$$\frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)}.$$

Equivalently, we can write this as the ordinary differential equation

$$\frac{dx}{dy} = -\frac{Q(x,y)}{P(x,y)}.$$

If $P(x,y)\,dx + Q(x,y)\,dy$ is an exact differential, and is equal to df, then we have $df = 0$ and solutions of the differential equation are given by $f(x,y) = const.$

Example C

Consider $y\,dx + x\,dy = 0$, which we have already shown to be an exact differential and equal to $d(xy) = 0$ giving solutions $xy = A$, where A is a constant, or equivalently $y = A/x$.

Alternatively, we could have rewritten the differential as $\frac{dy}{dx} = -\frac{y}{x}$ and used separation of variables $\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x} \Rightarrow \ln y = -\ln x + c \Rightarrow$ $y = A/x$.

2.2.6 Integrating factors

In §1.2.5 we introduced the idea of an *integrating factor* for an ordinary differential equation. Here we extend this idea to differentials.

One way of making use of this idea of exact differentials for equations that are not exact is to find an *integrating factor* that transforms the equation into an exact one.

Consider the differential $P(x,y)\,dx + Q(x,y)\,dy$ and the function $\mu(x,y)$. This function is the *integrating factor* for the differential if

$$\mu(x,y)\,P(x,y)\,dx + \mu(x,y)\,Q(x,y)\,dy$$

is exact, *i.e.* there exists some f for which

$$df = \mu P\,dx + \mu Q\,dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

This requires

$$\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q)$$

$$\Rightarrow \qquad \mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) + P\frac{\partial \mu}{\partial y} - Q\frac{\partial \mu}{\partial x} = 0.$$

This *partial differential equation* for μ is normally very difficult to solve (and hence of not much value). However, a special case is when there exists $\mu = \mu(x)$ (so that $\partial\mu/\partial y = 0$) and so

$$\mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) - Q\frac{d\mu}{dx} = 0$$

$$\Rightarrow \qquad \frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

which is self-consistent provided the right-hand side is a function only of x. In such cases,

$$\mu = \exp\left(\int \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dx\right)$$

Similarly, when there is an integrating factor $\mu(y)$ it is the solution of

$$\frac{1}{\mu}\frac{d\mu}{dy} = -\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

if the right-hand side is a function only of y.

It should be stressed that we cannot always find $\mu(x)$ or $\mu(y)$, but when we can it will generally make the solution of the equation easier.

Example A

Find the integrating factor $\mu(x)$ that makes

$$[\cot x \sin(x + y) + \cos(x + y)]dx + \cos(x + y)\, dy$$

exact.

$$\Rightarrow \qquad \frac{\partial P}{\partial y} = \cot x \cos\left(x+y\right) - \sin\left(x+y\right)$$

$$\frac{\partial Q}{\partial x} = -\sin\left(x+y\right).$$

Since these are not equal, the differential is not exact. However,

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = \cot x \cos\left(x+y\right) = Q\cot x$$

suggesting an integrating factor that is the solution of

$$\frac{1}{\mu}\frac{d\mu}{dx} = \cot x = \frac{\cos x}{\sin x} \quad \Rightarrow \quad \int \frac{d\mu}{\mu} = \int \frac{d\left(\sin x\right)}{\sin x}$$

$$\Rightarrow \qquad \mu = \sin x.$$

[The arbitrary multiplicative constant does not affect the use of the integrating factor and so can be dropped.]

We can check this by substituting

$$\begin{aligned}\mu P dx + \mu Q dy &= \sin x\left(\cot x \sin\left(x+y\right) + \cos\left(x+y\right)\right)dx \\ &\quad + \sin x \cos\left(x+y\right)dy \\ &= \left(\cos x \sin\left(x+y\right) + \sin x \cos\left(x+y\right)\right)dx \\ &\quad + \sin x \cos\left(x+y\right)dy \\ &= d\left(\sin x \sin\left(x+y\right)\right)\end{aligned}$$

Note, had we looked for an integrating factor $\eta(y)$, we would have found

$$\frac{1}{\eta}\frac{d\eta}{dy}=-\frac{1}{P}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=-\frac{\cot x\ \cos(x+y)}{\cot x\ \sin(x+y)+\cos(x+y)}.$$

As the right-hand side is an irreducible function of both x and y, there is no integrating factor of the form $\eta(y)$.

Example B

Earlier we saw $y\,dx - x\,dy$

was not an exact differential. Here we have

$$\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}=2=\frac{2}{y}P=-\frac{2}{x}Q$$

so we can introduce an integrating factor either as $\mu(x)$ or as $\mu(y)$. Taking $\mu(x)$, we have

$$\frac{1}{\mu}\frac{d\mu}{dx}=\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=-\frac{2}{x}$$

$\Rightarrow$ $\ln\mu = -2\ln x = \ln x^{-2} \quad \Rightarrow \quad \mu = x^{-2}.$

Checking: $\mu P dx+\mu Q dy=\frac{y}{x^2}dx-\frac{dy}{x}=-d\left(\frac{y}{x}\right)$

In the Example B it was obvious that there was more than one possible integrating factor. Indeed, in general, the integrating factor will not be unique.

Example C

In Examples Sheet 1 you were asked to solve

$$(\ln y - x)\frac{dy}{dx} - y\ln y = 0.$$

You might have elected to solve this using the substitution

$$u = \ln y - x$$

$\Rightarrow$ $$y = e^{u+x}, \quad \frac{dy}{dx} = e^{u+x}\left(\frac{du}{dx}+1\right)$$

$\Rightarrow$ $$u\left[e^{u+x}\left(\frac{du}{dx}+1\right)\right] - e^{u+x}(u+x) = 0$$

$\Rightarrow$ $$u\frac{du}{dx} - x = 0$$

$\Rightarrow$ $$u^2 = x^2 + c$$

$\Rightarrow$ $$(\ln y - x)^2 = (\ln y)^2 - 2x\ln y + x^2 = x^2 + c$$

$\Rightarrow$ $$x = \frac{1}{2}\left(\ln y + \frac{c}{\ln y}\right).$$

We now have the framework to handle this equation in a different way that does not require guessing the correct substitution.

Solving via integrating factor

Rewriting the equation in the standard form for a differential

$$(y\ln y)dx + (x - \ln y)dy = 0.$$

Now $P = y \ln y$ and $Q = x - \ln y$. Testing

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = (\ln y + 1) - 1 = \ln y$$

is not exact, so we look for an integrating factor. Since the above is a function only of y, it makes sense to look for an integrating factor of the form $\mu(y)$, and so we need to solve

$$\frac{1}{\mu}\frac{d\mu}{dy} = -\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = -\frac{1}{y\ln y}(\ln y) = -\frac{1}{y}$$

thus $$\ln\mu = c - \ln y$$

$\Rightarrow$ $$\mu = A/y.$$

As this is an integrating factor, we can take the constant $A = e^c = 1$ without loss of generality. The new differential is then

$$\begin{aligned} df &= \frac{1}{y}\left[(y\ln y)dx + (x - \ln y)dy\right] \\ &= \ln y\ dx + \left(\frac{x - \ln y}{y}\right)dy \\ &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\ &= 0 \end{aligned}$$

Considering the partial derivatives

$$\frac{\partial f}{\partial x} = \ln y \quad \rightarrow \quad f(x,y) = x\ln y + g(y),$$

$$\frac{\partial f}{\partial y} = \frac{x - \ln y}{y} \quad \rightarrow \quad f(x,y) = x\ln y - \frac{1}{2}(\ln y)^2 + h(x)$$

Comparing gives

$$f(x,y) = x\ln y - \frac{1}{2}(\ln y)^2 + b = 0$$

that can be rearranged to

$$x = \frac{1}{2}\left(\ln y + \frac{c}{\ln y}\right)$$

as before, if we take the arbitrary constant $b = -\tfrac{1}{2}c$.

A third way of solving this equation

Rewrite the equation in terms of *dx*/*dy* instead of *dy*/*dx*:

$$(\ln y - x) - y\ln y \frac{dx}{dy} = 0$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\ln y}{y\ln y} - \frac{x}{y\ln y} = \frac{1}{y} - \frac{x}{y\ln y}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{1}{y\ln y}x = \frac{1}{y}$$

which is linear with $p(y) = 1/(y \ln y)$. The integrating factor is

$$I(y) = \exp\left(\int p(y)\,dy\right) = \exp\left(\int \frac{dy}{y\ln y}\right) = \exp(\ln(\ln y)) = \ln y$$

so

$$\frac{d}{dy}\left(I(y)x\right) = \frac{I(y)}{y} = \frac{\ln y}{y}$$

$$\Rightarrow \quad I(y)x = x\ln y = \int \frac{\ln y}{y}dy = \frac{1}{2}(\ln y)^2 + b$$

thus

$$x = \frac{1}{2}\left(\ln y + \frac{c}{\ln y}\right)$$

Relationship with integrating factors for linear odes
(You do not need to be able to replicate this)

In §1.2.5 we introduced the idea of an *integrating factor* for a first-order linear equation. The integrating functions discussed here can be viewed as a generalisation of this approach. When we introduced integrating factors we were considering equations of the form

$$\frac{dy}{dx} + p(x)y = f(x).$$

Rearranging this as a differential,

$$\left(p(x)y - f(x)\right)dx + dy = 0$$

Gives us $P = p(x)y - f(x)$ and $Q = 1$. Given that $\partial P/\partial y = p(x)$ and $\partial Q/\partial x = 0$ it makes sense to look for an integrating function of the form $\mu(x)$, so

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= \frac{1}{1}\big(p(x) - 0\big) = p(x)$$

$$\Rightarrow \qquad \ln \mu = \int p(x)\,dx$$

$$\Rightarrow \qquad \mu = \exp\left(\int p(x)\,dx\right)$$

is exactly the integrating factor introduced previously.

2.2.7 Examples of differentiation and differentials

2002 Paper 2

Consider the change of variables

$$x = e^{-s}\sin t,\ \ y = e^{-s}\cos t \text{ such that } u(x,y) = v(s,t).$$

(a) Use the chain rule to express $\partial v/\partial s$ and $\partial v/\partial t$ in terms of x, y, $\partial u/\partial x$ and $\partial u/\partial y$.

(b) Find, similarly, an expression for $\partial^2 v/\partial t^2$.

(c) Hence transform the equation

$$y^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial y^2} = 0$$

into a partial differential equation for v.

Note: This example introduces the idea of a *partial differential equation*. We shall consider such equations in greater depth in §2.4. However, we already have all the ideas necessary to tackle this question.

Solution to (a)

The chain rule gives

$$\frac{\partial v}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$= -e^{-s}\sin t\frac{\partial u}{\partial x} - e^{-s}\cos t\frac{\partial u}{\partial y}$$

$$= -x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}$$

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$= e^{-s}\cos t\frac{\partial u}{\partial x} - e^{-s}\sin t\frac{\partial u}{\partial y}$$

$$= y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y}$$

Solution to (b)

Find $\partial^2 v/\partial t^2$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial t}\right)=\frac{\partial x}{\partial t}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right)+\frac{\partial y}{\partial t}\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial t}\right)$$

$$=\frac{\partial x}{\partial t}\frac{\partial}{\partial x}\left(y\frac{\partial u}{\partial x}-x\frac{\partial u}{\partial y}\right)+\frac{\partial y}{\partial t}\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial x}-x\frac{\partial u}{\partial y}\right)$$

$$=e^{-s}\cos t\left(y\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial y}-x\frac{\partial^2 u}{\partial x\partial y}\right)-e^{-s}\sin t\left(\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}-x\frac{\partial^2 u}{\partial y^2}\right)$$

$$=y\left(y\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial y}-x\frac{\partial^2 u}{\partial x\partial y}\right)-x\left(\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}-x\frac{\partial^2 u}{\partial y^2}\right)$$

$$=y^2\frac{\partial^2 u}{\partial x^2}-y\frac{\partial u}{\partial y}-xy\frac{\partial^2 u}{\partial x\partial y}-x\frac{\partial u}{\partial x}-xy\frac{\partial^2 u}{\partial x\partial y}+x^2\frac{\partial^2 u}{\partial y^2}$$

$$=y^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+x^2\frac{\partial^2 u}{\partial y^2}-y\frac{\partial u}{\partial y}-x\frac{\partial u}{\partial x}$$

Solution to (c)

Comparing the differential equation

$$y^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+x^2\frac{\partial^2 u}{\partial y^2}=0$$

with $$\frac{\partial^2 v}{\partial t^2}=y^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+x^2\frac{\partial^2 u}{\partial y^2}-\left(y\frac{\partial u}{\partial y}+x\frac{\partial u}{\partial x}\right)$$

and noting that the last bracketed term is $\partial v/\partial s$, then the differential equation may be written as

$$y^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+x^2\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 v}{\partial t^2}-\frac{\partial v}{\partial s}=0.$$

This transformed equation is simpler than the original: it has fewer terms! We could find a solution to this easily by writing $v = S(s)\,T(t)$ and substituting: you do not need to be able to do this for NST1A.

2000 Paper 2

(a) Give a necessary condition for the differential

$$P(x,y)\,dx + Q(x,y)\,dy$$

to be exact.

Show that

$$w = \left[1 - y\exp\left\{\frac{y}{x+y}\right\}\right]dx + \left[1 + x\exp\left\{\frac{y}{x+y}\right\}\right]dy$$

is not exact.

(b) Let

$$x + y = u,$$

$$y = uv.$$

Express dx and dy in terms of du and dv.

Hence express w in terms of u, v, du and dv.

Find an integrating factor, μ, in terms of u and v such that μw is exact.

Hence solve $w = 0$, expressing your answer in terms of x and y.

Solution to (a)

$\partial P/\partial y = \partial Q/\partial x$ is a necessary condition for the differential to be exact.

For the differential,

$$w = \left[1 - y\exp\left\{\frac{y}{x+y}\right\}\right]dx + \left[1 + x\exp\left\{\frac{y}{x+y}\right\}\right]dy$$

we have

$$\frac{\partial P}{\partial y} = -\exp\left\{\frac{y}{x+y}\right\} - y\left(\frac{x+y-y}{(x+y)^2}\right)\exp\left\{\frac{y}{x+y}\right\}$$

$$= -\left(1 + \frac{xy}{(x+y)^2}\right)\exp\left\{\frac{y}{x+y}\right\}$$

$$= -\frac{x^2+3xy+y^2}{(x+y)^2}\exp\left\{\frac{y}{x+y}\right\}$$

$$\frac{\partial Q}{\partial x} = \exp\left\{\frac{y}{x+y}\right\} - x\left(\frac{y}{(x+y)^2}\right)\exp\left\{\frac{y}{x+y}\right\}$$

$$= \frac{x^2+xy+y^2}{(x+y)^2}\exp\left\{\frac{y}{x+y}\right\}$$

Since $\partial P/\partial y \neq \partial Q/\partial x$ then w is not exact.

Solution to (b)

We have $x + y = u$ and $y = uv \Rightarrow x = u - uv = u(1-v)$

The differentials

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$$

$$= (1-v)du - udv$$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$

$$= vdu + udv$$

$$w=\left[1-y\exp\left\{\frac{y}{x+y}\right\}\right]dx+\left[1+x\exp\left\{\frac{y}{x+y}\right\}\right]dy$$

$$=\left[1-uv\exp\left\{\frac{uv}{u}\right\}\right]\left[(1-v)du-udv\right]$$

$$+\left[1+u(1-v)\exp\left\{\frac{uv}{u}\right\}\right]\left[vdu+udv\right]$$

$$=\left[(1-v)\left(1-uve^{v}\right)+v\left(1+u(1-v)e^{v}\right)\right]du$$

$$+\left[-u\left(1-uve^{v}\right)+u\left(1+u(1-v)e^{v}\right)\right]dv$$

$$=\left[1-v-uve^{v}+uv^{2}e^{v}+v+uve^{v}-uv^{2}e^{v}\right]du$$

$$+\left[-u+u^{2}ve^{v}+u+u^{2}e^{v}-u^{2}ve^{v}\right]dv$$

$$=du+u^{2}e^{v}dv$$

Now need to find an integrating factor μ such that

$$\mu\, du + \mu\, u^2e^v\, dv$$

is exact. This requires $\partial P/\partial v = \partial Q/\partial u = \partial\mu/\partial v = u^2e^v\partial\mu/\partial u + 2\mu ue^v$. If we select $\mu = \mu(u)$, then

$$\frac{1}{\mu}\frac{d\mu}{du}=-\frac{2}{u} \quad\Rightarrow\quad \ln\mu = \ln u^{-2}$$

$\Rightarrow$

$$\mu = u^{-2}$$

and $$\mu w=\mu du+\mu u^{2}e^{v}dv=\frac{1}{u^{2}}du+e^{v}dv=d\left(-\frac{1}{u}+e^{v}\right)$$

is exact.

Setting $w = 0$ implies that $\mu w = 0$ so $d\left(-\frac{1}{u}+e^{v}\right)=0$ and

$$-\frac{1}{u}+e^{v}=-\frac{1}{x+y}+\exp\left(\frac{y}{x+y}\right)=const$$

2001 Paper 2

Give a necessary condition for the expression

$$P(x,y)\,dx + Q(x,y)\,dy$$

to be an exact differential.

For the thermodynamics of a gas, the internal energy U can be regarded as a function of the entropy S and the volume V. It is given that

$$dU = TdS - pdV$$

where T is the temperature and p the pressure. By considering the function

$$A = U - TS,$$

or by some other method, show that

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V .$$

Now, considering U as a function of T and V, show that

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - p .$$

Given

$$p = \frac{nRT}{V - nb}\exp\left\{-\frac{an}{VRT}\right\},$$

where a, b, n, R are constants, find $\left(\frac{\partial U}{\partial V}\right)_T$. If, instead,

$$p = \frac{nRT}{V}$$

and $\left(\frac{\partial U}{\partial T}\right)_V = C_V$, where C_V is constant, find an expression for U.

Solution

(a) For the equation to be exact, $\partial P/\partial y = \partial Q/\partial x$.

(b) The differential of $A = U - TS$ is

$$dA = dU - TdS - SdT$$
$$= TdS - pdV - TdS - SdT$$
$$= -pdV - SdT$$

given $dU = TdS - pDV$

As this differential is exact, we require

$$\left(\frac{\partial p}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T.$$

(c) Taking U as a function of T and V then

$$U = U(T,V) \Rightarrow S = S(T,V)$$
$$dS = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV$$

$$dU = TdS - pdV$$
$$= T\left(\left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV\right) - pdV$$
$$= T\left(\frac{\partial S}{\partial T}\right)_V dT + \left(T\left(\frac{\partial S}{\partial V}\right)_T - p\right)dV$$
$$= \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV$$

hence $dV \Rightarrow$ $$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - p$$

and $dT \Rightarrow$ $$\left(\frac{\partial U}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V.$$

(d) From the earlier result (b) we have $\left(\frac{\partial p}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T$, and have been given $p = p(T,V)$, so from (c)

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - p = T\left(\frac{\partial p}{\partial T}\right)_V - p$$

Now if $p = \frac{nRT}{V - nb}\exp\left\{-\frac{an}{VRT}\right\}$, then

$$\left(\frac{\partial p}{\partial T}\right)_V = \frac{nR}{V-nb}\exp\left\{-\frac{an}{VRT}\right\} + \frac{nRT}{V-nb}\left(\frac{an}{VRT^2}\right)\exp\left\{-\frac{an}{VRT}\right\}$$

$$= \frac{n(VRT+an)}{VT(V-nb)}\exp\left\{-\frac{an}{VRT}\right\}$$

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p$$

$$= \left[\frac{n(VRT+an)}{V(V-nb)} - \frac{nRT}{V-nb}\right]\exp\left\{-\frac{an}{VRT}\right\}$$

$$= \frac{n(VRT+an)-nVRT}{V(V-nb)}\exp\left\{-\frac{an}{VRT}\right\}$$

$$= \frac{an^2}{V(V-nb)}\exp\left\{-\frac{an}{VRT}\right\}$$

(e) The final part of the solution has $p = \frac{nRT}{V}$ and $\left(\frac{\partial U}{\partial T}\right)_V = C_V$ constant. Hence,

$$U = U(T,V) \Rightarrow dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV = C_v dT + \left(\frac{\partial U}{\partial V}\right)_T dV$$

$$\Rightarrow \qquad U = C_v T + f(V),$$

where $f(V)$ is arbitrary. However, in (c) we had

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - p = T\left(\frac{\partial p}{\partial T}\right)_V - p$$

but also we now have that $p = \frac{nRT}{V}$ so

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p$$

$$= T\left(\frac{nR}{V}\right) - \frac{nRT}{V}$$

$$= 0$$

This leads to the result that $U = C_V T + const.$ since we must have $\partial U/\partial V = \partial f/\partial V = 0$.

2.3 Stationary points

2.3.1 Stationary points with one independent variable

For functions of a single variable, a stationary point is normally a *maximum* or a *minimum*, although in some circumstances can be a *point of inflection*.

Note that there is generally a distinction between the *local maximum* and the *global maximum*, and likewise for the *local minimum* and *global minimum*.

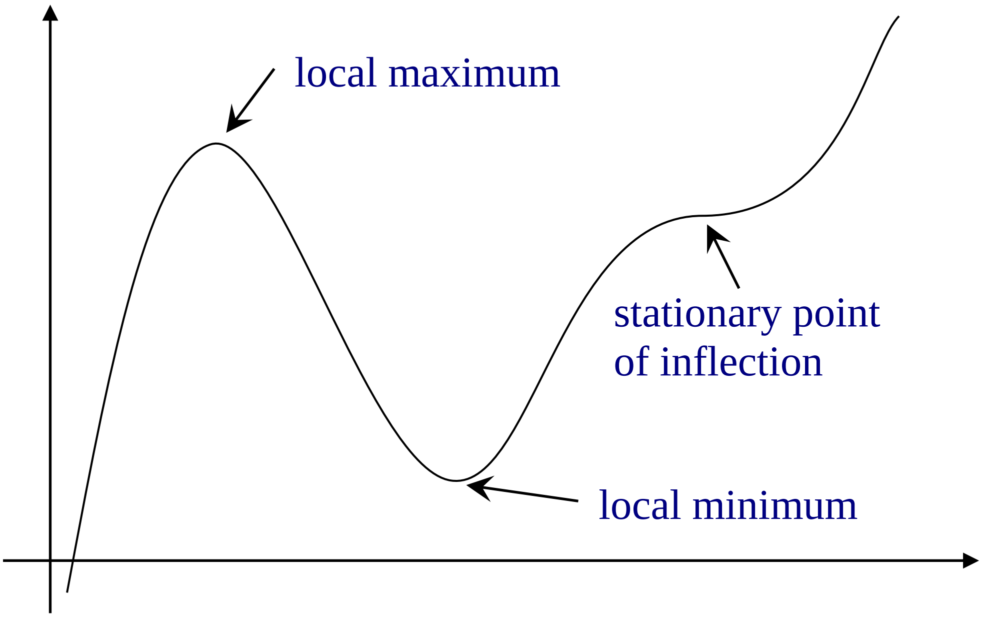


Figure 8: Stationary points of functions of a single variable

As we have seen, near any point x_0 we may approximate a function $f(x)$ by a Taylor series expansion. In order to understand the nature of the stationary point, we need to include to at least the quadratic term in the series,

$$f(x) \approx f(x_0) + (x - x_0)f'(x_0) + ½(x - x_0)^2 f''(x_0),$$

thus approximating the function by a parabola.

A *local maximum* at some point $x = x_0$ has the property that $f(x) \le f(x_0)$ for all x sufficiently close to x_0, *i.e.*

$$f(x_0) + (x - x_0)f'(x_0) + ½(x - x_0)^2 f''(x_0) \le f(x_0)$$

$$\Rightarrow \qquad (x - x_0)f'(x_0) + ½(x - x_0)^2 f''(x_0) \le 0.$$

When $x - x_0$ is small, then $(x - x_0)^2$ is even smaller, so the first term dominates close to x_0 and thus the condition can only be satisfied if $f'(x_0) = 0$. Hence we must have

$$\Rightarrow \qquad ½(x - x_0)^2 f''(x_0) \le 0,$$

which clearly requires $f''(x_0) \le 0$.

Similarly, for a *local minimum* we have $f(x) \ge f(x_0)$ for all x sufficiently close to x_0 requiring $f'(x_0) = 0$ and $f''(x_0) \ge 0$.

Vanishing second derivative

A stationary point $f'(x_0) = 0$ at which $f''(x_0) = 0$ satisfies the conditions for both a local maximum (*i.e.* $f''(x_0) \le 0$) and a local minimum (*i.e.* $f''(x_0) \ge 0$). The conditions are only *sufficient* when there is inequality.

To determine the nature of the stationary point when $f''(x_0) = 0$ we must look to higher derivatives, effectively approximating $f(x)$ by a higher order polynomial.

If the first non-zero derivative is $f'''(x_0)$, then

$$f(x) \approx f(x_0) + \tfrac{1}{6}(x - x_0)^3 f'''(x_0)$$

and the stationary point is a *stationary point of inflection*.

If the first non-zero derivative is $f^{iv}(x_0)$, then

$$f(x) \approx f(x_0) + \tfrac{1}{24}(x - x_0)^4 f^{iv}(x_0)$$

and the stationary point is a *local maximum* if $f^{iv}(x_0) < 0$ or a *local minimum* if $f^{iv}(x_0) > 0$.

These ideas extend further so that if the first non-zero derivative is *odd* then we have a point of inflection (increasing towards higher x if the derivative is positive). If the first non-zero derivative is even then we have a local maximum if it is negative, or a local minimum if it is positive.

2.3.2 Stationary points with more than one independent variable

We shall focus on functions of two independent variables.

The linear terms of Taylor series expansion of $f(x,y)$ near the point (x_0,y_0) can be written as

$$f(x_0 + \delta x, y_0 + \delta y) \approx f(x_0, y_0) + \delta x \frac{\partial f}{\partial x}(x_0, y_0) + \delta y \frac{\partial f}{\partial y}(x_0, y_0).$$

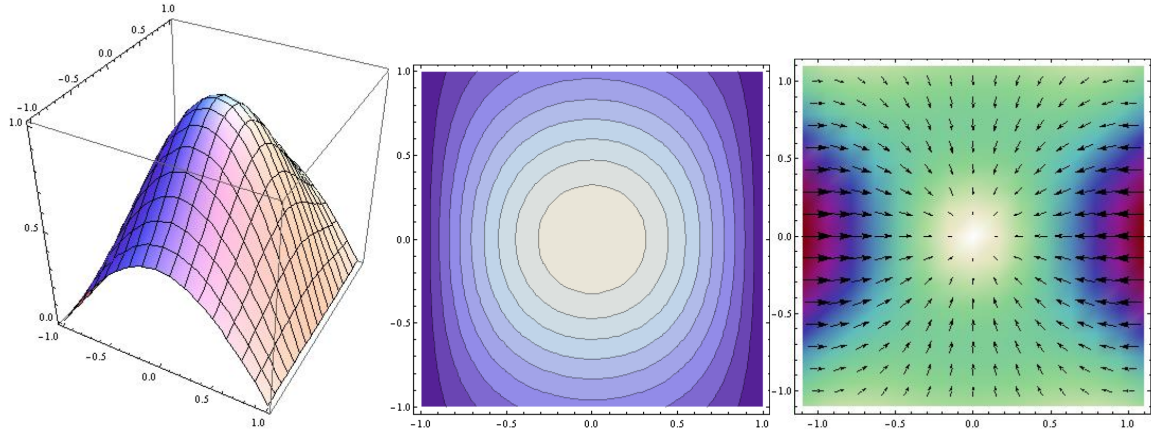
The error in this approximation is quadratic in δx and δy, corresponding to the higher-order terms in the Taylor series. [We shall look at this later.]

The point (x_0,y_0) is said to be *stationary* if the linear terms vanish for all δx and δy, *i.e.*

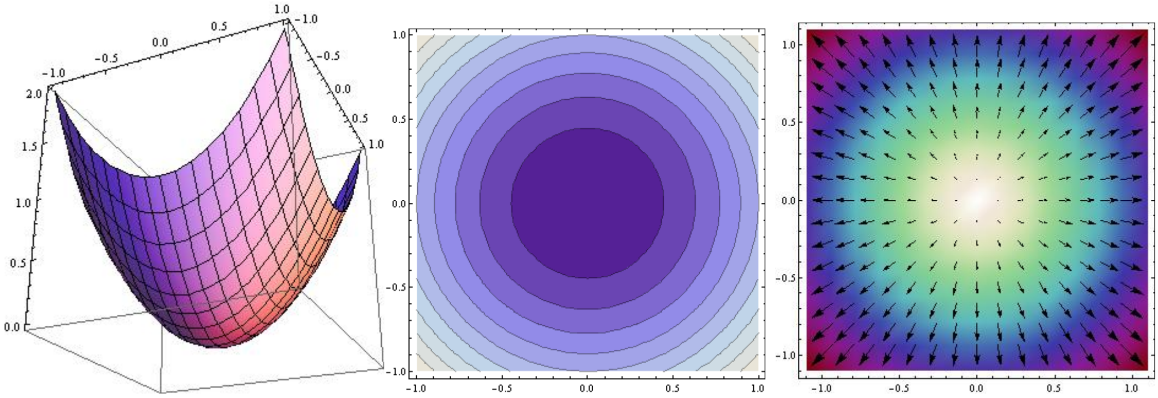
$$\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0, y_0) = f_y(x_0, y_0) = 0.$$

This is a generalisation of the condition $df/dx = 0$ for functions of a single variable.

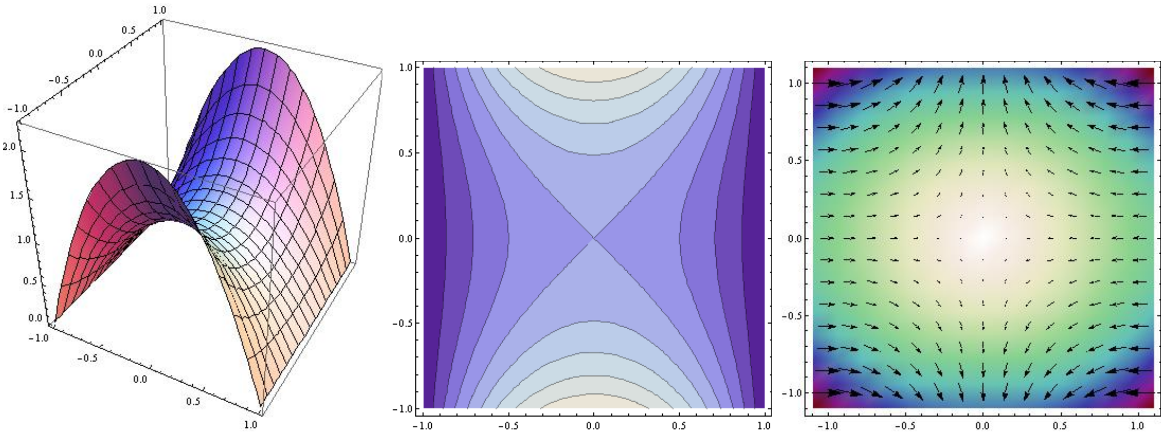
If we visualise the function as the surface $z = f(x,y)$, then the surface looks *horizontal* close to (x_0,y_0).



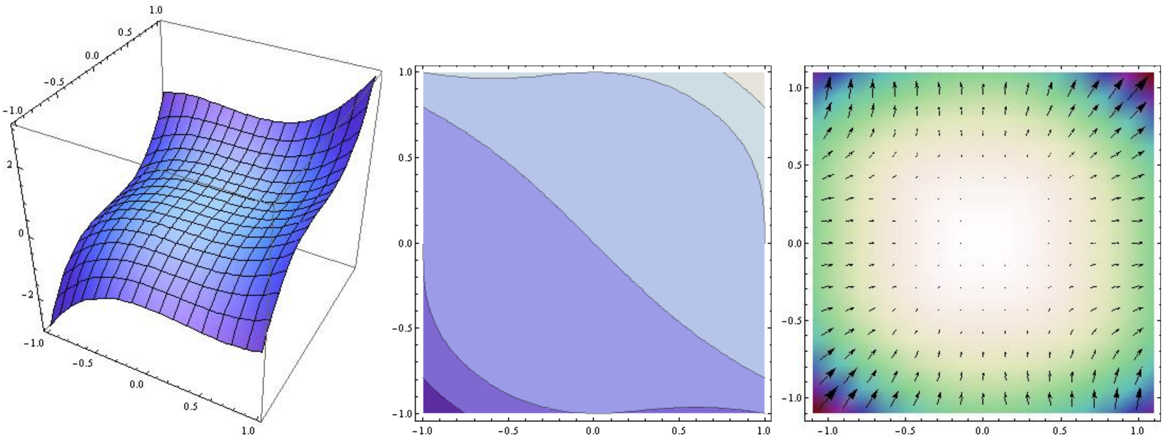
(a) $f(x,y) = (1 - x^2)\exp(-y^2)$



(b) $f(x,y) = x^2 + y^2$



(c) $f(x,y) = (1 - x^2)\exp(y^2/(1 + y^2/4))$



(d) $f(x,y) = x^3 + (1 + x^2)y^3$

Figure 9: Stationary points with two independent variables. (a) Local maximum, (b) local minimum, (c) saddle point and (d) point of inflection. Left-hand column shows surface. Central column shows contour plot. Right-hand column shows gradient vector superimposed on a colour scale related to the magnitude of the gradient.

In addition to stationary ‘points’, ridges and valleys can represent higher-dimensional regions over which the function is constant. We shall not, however, focus on such features here.

We can generalise the condition for a stationary point using the definition of the *gradient vector* we introduced in §2.2.1 and noting that the condition that *all* first derivatives vanish is $\nabla f = \mathbf{0}$. For a function of n variables, this indicates that all n first derivatives simultaneously vanish. [Note that $\mathbf{0}$ is a vector of n zeros, not a single zero since ∇f is itself a vector of n elements. Warning: some examiners might penalise you if you make the statement that a vector equals a scalar!]

Using this gradient notation, we can write the approximation for $f(x,y) = f(\mathbf{x})$ as

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \delta\mathbf{x}\cdot\nabla f,$$

where $\delta\mathbf{x} = \mathbf{x} - \mathbf{x}_0$ is the vector between the point $\mathbf{x}$ (given as a vector from the origin) and the point $\mathbf{x}_0$.

Type of stationary point

When we had only one independent variable, the second derivative gave us information about the nature of the stationary point: $f'' < 0$ implies a local maximum, $f'' > 0$ a local minimum, while for $f'' = 0$ we needed to look to higher derivatives.

By analogy, for two independent variables, we choose a quadratic form

$$f(x,y) \approx \hat{f}(x,y) \equiv f(x_0,y_0) + A(x-x_0) + B(y-y_0) + C(x-x_0)^2 + D(x-x_0)(y-y_0) + E(y-y_0)^2$$

where A, B, C, D and E are constants. We choose these constants so that at the point (x_0,y_0) the value of the function, the first and second derivatives all match exactly.

For the first derivatives,

$$\frac{\partial \hat{f}}{\partial x} = A + 2C(x-x_0) + D(y-y_0) \text{ must equal } \frac{\partial f}{\partial x} \text{ at } (x_0,y_0)$$

$$\Rightarrow \qquad A = \frac{\partial f}{\partial x}(x_0,y_0).$$

Similarly,

$$B = \frac{\partial f}{\partial y}(x_0,y_0).$$

For the second derivatives, we have

$$\frac{\partial^2 \hat{f}}{\partial x^2} = \frac{\partial}{\partial x}\left[A + 2C(x-x_0) + D(y-y_0)\right] = 2C$$

matching $\partial^2 f/\partial x^2$ at (x_0,y_0)

$$\frac{\partial^2 \hat{f}}{\partial x \partial y} = \frac{\partial}{\partial y}\left[A + 2C(x-x_0) + D(y-y_0)\right] = D$$

matching $\partial^2 f/\partial x \partial y$ at (x_0,y_0)

$$\frac{\partial^2 \hat{f}}{\partial y^2} = \frac{\partial}{\partial y}\left[B + D(x - x_0) + 2E(y - y_0)\right] = 2E$$

matching $\partial^2 f/\partial y^2$ at (x_0,y_0)

$$\Rightarrow \qquad C = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}, \quad D = \frac{\partial^2 f}{\partial x \partial y}, \quad E = \frac{1}{2}\frac{\partial^2 f}{\partial y^2}$$

The behaviour of the function near (x_0,y_0) is therefore

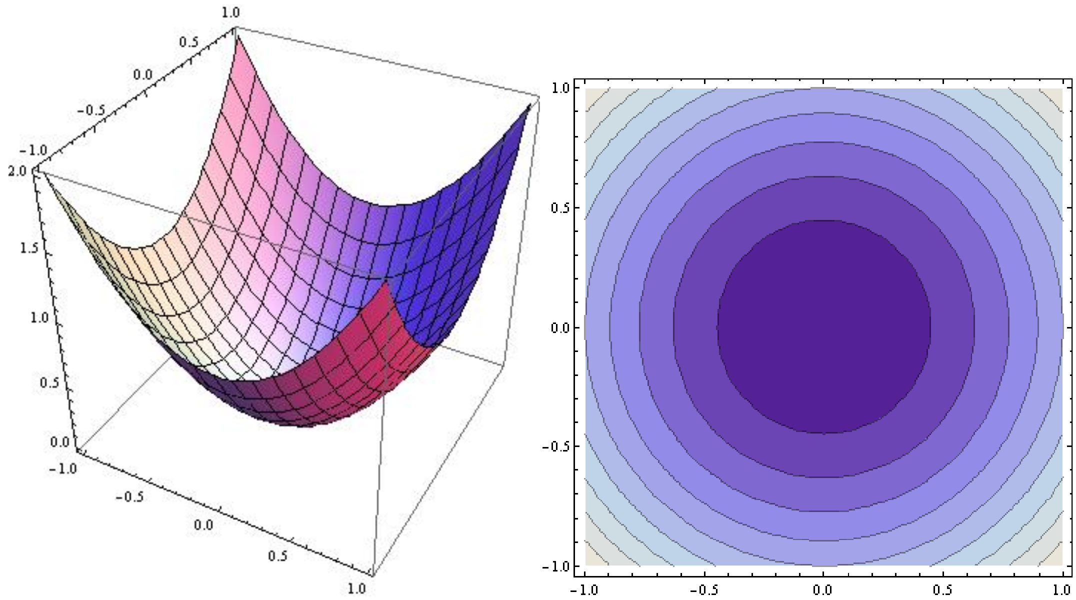
$$\begin{aligned} f(x,y) \approx{}& f(x_0,y_0) + (x - x_0)\frac{\partial f}{\partial x}(x_0,y_0) + (y - y_0)\frac{\partial f}{\partial y}(x_0,y_0) \\ &+ \frac{1}{2}(x - x_0)^2 \frac{\partial^2 f}{\partial x^2}(x_0,y_0) \\ &+ (x - x_0)(y - y_0)\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) \\ &+ \frac{1}{2}(y - y_0)^2 \frac{\partial^2 f}{\partial y^2}(x_0,y_0) \end{aligned}$$

These are the first few terms of a Taylor series expansion of $f(x,y)$.

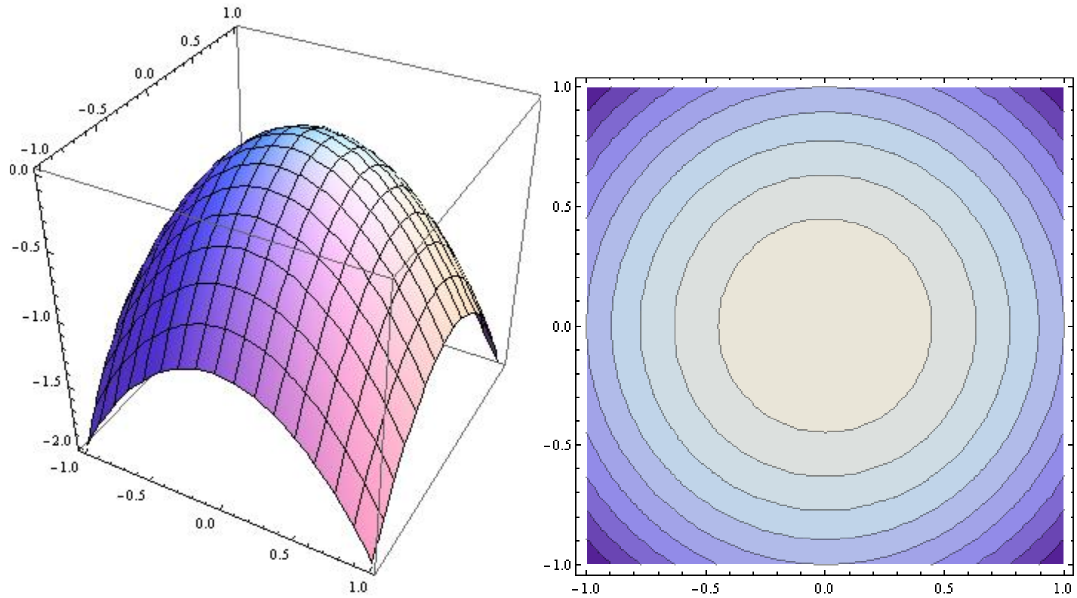
If we consider a stationary point at (x_0,y_0), then we know that $\partial f/\partial x$ and $\partial f/\partial y$ vanish, so near the stationary point the function behaves as the quadratic

$$f(x,y) \approx f(x_0,y_0) + \tfrac{1}{2}(x - x_0)^2 f_{xx}(x_0,y_0) + (x - x_0)(y - y_0) f_{xy}(x_0,y_0) + \tfrac{1}{2}(y - y_0)^2 f_{yy}(x_0,y_0).$$

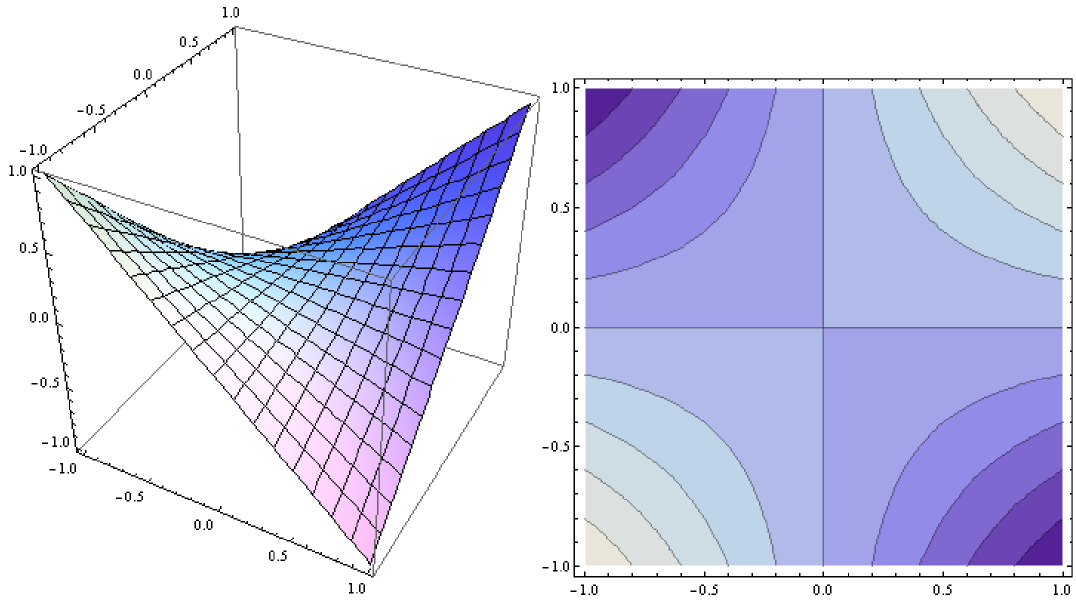
Consider now the following canonical forms, each with a stationary point at (0,0):



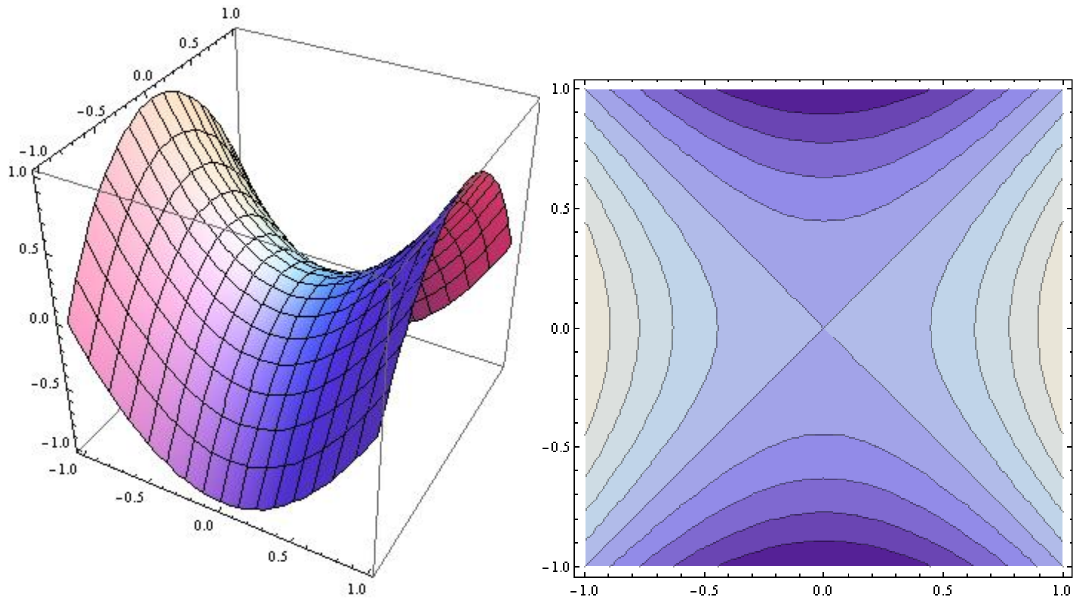
(a) $f(x,y) = x^2 + y^2$.



(b) $f(x,y) = -x^2 - y^2$.



(c) $f(x,y) = xy$.



(d) $f(x,y) = x^2 - y^2$.

As we saw previously, for functions of one variable, there are two ‘common’ types of stationary point: local minima and local maxima. (Stationary points of inflection, valleys and ridges are much less common.)

For functions of two (or more) variables, there are three common types of stationary points: local minima (see (a)), local maxima (see (b)) and *saddle points* (see (c) and (d)).

2.3.3 Classification of stationary points

Given a stationary point at (x_0,y_0), for a function $f(x,y)$ of two variables, how do we classify the type of stationary point?

Consider the (local) quadratic

$$Q(\delta x,\delta y)=\frac{1}{2}\delta x^2 f_{xx}+\delta x\delta y f_{xy}+\frac{1}{2}\delta y^2 f_{yy},$$

where $x = x_0 + \delta x$, $y = y_0 + \delta y$, and the partial derivatives are evaluated at (x_0,y_0).

As we move away from (x_0,y_0) the quadratic $Q(\delta x,\delta y)$ can behave in one of three possible ways:

(a) $Q(\delta x,\delta y)$ is positive for all choices of $\delta x,\delta y$ (except (0,0)). Here, $f(x,y)$ increases in all directions away from (x_0,y_0) and so $f(x_0,y_0)$ is a local minimum.

(b) $Q(\delta x,\delta y)$ is negative for all choices of $\delta x,\delta y$ (except (0,0)). Here, $f(x,y)$ decreases in all directions away from (x_0,y_0) and so $f(x_0,y_0)$ is a local maximum.

(c) $Q(\delta x,\delta y)$ is positive for some choices of $\delta x,\delta y$ and negative for others. Hence $f(x,y)$ increases in some directions but decreases in others. This is a *saddle point*.

Condition–

For a **saddle point**, there must be some real nonzero values of $\delta x, \delta y$ for which $Q(\delta x, \delta y) > 0$ and others for which $Q(\delta x, \delta y) < 0$.

- Hence, since the function is continuous, there must also be real nonzero values of $\delta x, \delta y$ for which $Q(\delta x, \delta y) = 0$. If we assume $\delta y = \lambda \delta x$ for such points, then

$$\begin{aligned} Q(\delta x, \delta y) &= \frac{1}{2}\delta x^2 f_{xx} + \delta x \delta y f_{xy} + \frac{1}{2}\delta y^2 f_{yy} \\ &= \frac{1}{2}\left(f_{xx} + 2\lambda f_{xy} + \lambda^2 f_{yy} \right)\delta x^2 \\ &= 0 \end{aligned}$$

  For this to have real roots (*i.e.*, the directions λ exist only if $\lambda \in \mathbb{R}$) then we require

$$f_{xx} f_{yy} - f_{xy}^2 < 0.$$

  Note that if f_{xx} and f_{yy} take opposite signs then it follows immediately that this condition is satisfied. There may, however, be cases where f_{xx} and f_{yy} have the same sign but we still have a saddle point.

If we cannot find real λ, then there are no directions (contours) along which $Q(\delta x, \delta y)$ height remains constant, and so (x_0, y_0) must be either a local minimum or a local maximum.

For a **local minimum**, we have

$$Q(\delta x, \delta y) = \frac{1}{2}\left(\delta x^2 f_{xx} + 2\delta x \delta y f_{xy} + \delta y^2 f_{yy}\right) > 0$$

(unless $\delta x = \delta y = 0$), hence

- taking $\delta y = 0 \Rightarrow Q(\delta x, \delta y) = \frac{1}{2}\delta x^2 f_{xx}$, which shows that $f_{xx} > 0$
- taking $\delta x = 0 \Rightarrow Q(\delta x, \delta y) = \frac{1}{2}\delta y^2 f_{yy}$, which shows that $f_{yy} > 0$
- and we need
  $\delta x^2 f_{xx} + 2\delta x\delta y\, f_{xy} + \delta y^2 f_{yy} = (\delta x + f_{xy}\delta y/f_{xx})^2 f_{xx} + \delta y^2(f_{yy} - f_{xy}^2/f_{xx})$
  to be positive for all $\delta x, \delta y$. This is the case only if $(f_{yy} - f_{xy}^2/f_{xx}) > 0$. Since $f_{xx} > 0$, it follows that we need $f_{xx}f_{yy} - f_{xy}^2 > 0$.

For a **local maximum**, we have

$$Q(\delta x, \delta y) = \frac{1}{2}\left(\delta x^2 f_{xx} + 2\delta x \delta y f_{xy} + \delta y^2 f_{yy}\right) < 0$$

(unless $\delta x = \delta y = 0$), hence

- taking $\delta y = 0$, shows that $f_{xx} < 0$
- taking $\delta x = 0$, shows that $f_{yy} < 0$
- and we need
  $\delta x^2 f_{xx} + 2\delta x\delta y\, f_{xy} + \delta y^2 f_{yy} = (\delta x + f_{xy}\delta y/f_{xx})^2 f_{xx} + \delta y^2(f_{yy} - f_{xy}^2/f_{xx})$
  to be negative for all $\delta x, \delta y$. This is the case only if $(f_{yy} - f_{xy}^2/f_{xx}) < 0$. Since $f_{xx} < 0$, it follows that we need $f_{xx}f_{yy} - f_{xy}^2 > 0$. [Note reversal of inequality since multiplying by the negative quantity f_{xx}.]

The need for higher derivatives

The special case $f_{xx}f_{yy} - f_{xy}^2 = 0$ corresponds to $d^2f/dx^2 = 0$ for a function of a single variable. Such a point could be a local maximum, a local minimum, a saddle point, or a stationary point of inflection. We would need to investigate this using more terms of the Taylor series. However, this is beyond the present course.

Topology

For continuous functions of one variable, between any two local maxima, we must have a local minimum. Similarly, between any two local minima we must have a local maximum.

The topology for functions of two variables is much richer and more complex. A good way of thinking about this is interpreting the function as the height on a topographic map and thinking about where there must be saddles, peaks and lakes. For example:

- There must be contours circling around extrema;
- There must be at least one saddle point between any to local extrema of the same kind;
- Contours passing through saddle points must either join back together (having passed around at least one extremum) or continue to infinity.

2.3.4 Stationary point examples

Example A

A cuboid has sides of length x, y and z. Given the volume is fixed to be V, find the stationary values of the area of one of the faces perpendicular to the z axis plus two times the area of one of the faces perpendicular to the x axis plus three times the area of one of the faces perpendicular to the y axis, *i.e.* find the stationary points of $xy + 2yz + 3zx$. Classify these points.

Since $V = xyz = const$, we may replace $z = V/xy$ and write the required sum as

$$A(x,y) = xy + 2y(V/xy) + 3(V/xy)x = xy + 2V/x + 3V/y.$$

The stationary points are where the partial derivatives

$$\frac{\partial A}{\partial x} = y - \frac{2V}{x^2} \quad \text{and} \quad \frac{\partial A}{\partial y} = x - \frac{3V}{y^2}$$

vanish. Hence substituting $y = 2V/x^2$ into $\partial A/\partial y = 0$ gives

$$x - \frac{3V}{y^2} = x - \frac{3V}{4V^2x^{-4}} = x\left(1 - \frac{3}{4V}x^3\right) = 0$$

so stationary points occur when $x = 0$ and $x = (4V/3)^{1/3}$.

The corresponding y values are

$x = 0$: $y \to \infty$,

$x = (4V/3)^{1/3}$: $y = \dfrac{2V}{x^2} = 2V\left(\dfrac{3}{4V}\right)^{2/3} = \left(\dfrac{9V}{2}\right)^{1/3}$.

Hence there is only one stationary point. The value of $A(x,y)$ for this point is

$$\begin{aligned} A(x,y) &= xy + \frac{2V}{x} + \frac{3V}{y} \\ &= \left(\frac{4V}{3}\right)^{1/3}\left(\frac{9V}{2}\right)^{1/3} + 2V\left(\frac{3}{4V}\right)^{1/3} + 3V\left(\frac{2}{9V}\right)^{1/3} \\ &= \left(6V^2\right)^{1/3} + \left(6V^2\right)^{1/3} + \left(6V^2\right)^{1/3} \\ &= 3\left(6V^2\right)^{1/3} \end{aligned}$$

The classification of the stationary point comes from its second derivatives:

$$\frac{\partial^2 A}{\partial x^2} = \frac{\partial}{\partial x}\left(y - \frac{2V}{x^2}\right) = \frac{4V}{x^3} = \frac{4V}{4V/3} = 3$$

$$\frac{\partial^2 A}{\partial y^2} = \frac{\partial}{\partial y}\left(x - \frac{3V}{y^2}\right) = \frac{6V}{y^3} = \frac{6V}{9V/2} = \frac{4}{3}$$

$$\frac{\partial^2 A}{\partial x \partial y} = \frac{\partial}{\partial y}\left(y - \frac{2V}{x^2}\right) = 1$$

Now $A_{xx}A_{yy} - A_{xy}^2 = 3(4/3) - 1 = 3 > 0$, along with both $A_{xx} > 0$ and $A_{yy} > 0$. Hence the stationary point is a minimum.

Example B

Find and classify the stationary points of $f(x,y) = x^3 - 3x^2 + 2xy - y^2$. Sketch the contours of f.

For stationary points, the derivatives vanish, hence

$$f_x = 3x^2 - 6x + 2y = 0$$

$$f_y = 2x - 2y = 0.$$

The second of these requires $y = x$ and so, substituting into the first

$$3x^2 - 6x + 2x = x(3x - 4) = 0$$

gives stationary points $(x,y) = (0,0)$and $(x,y) = (4/3,4/3)$.

The second derivatives are

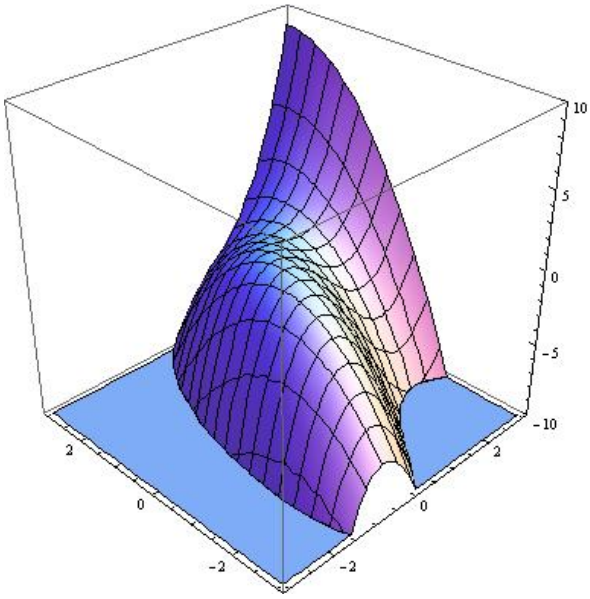
$$f_{xx} = 6x - 6,$$

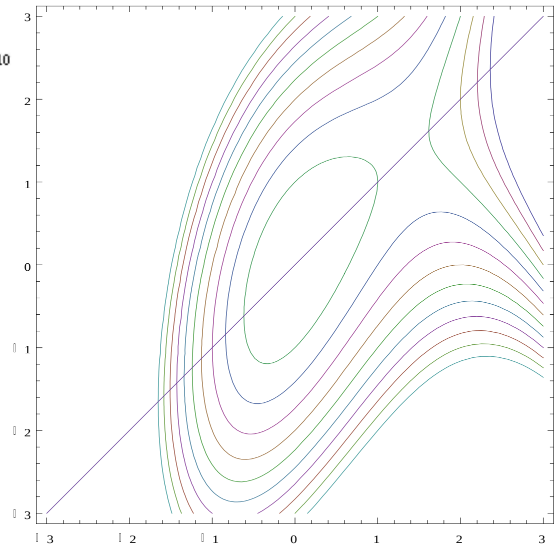
$$f_{xy} = 2,$$

$$f_{yy} = -2.$$

Hence, at $(0,0)$, we have $f_{xx}f_{yy} - f_{xy}^2 = -6\times(-2) - 2^2 = 8 > 0$ and so we have a local maximum.

At $(4/3,4/3)$ we have $f_{xx}f_{yy} - f_{xy}^2 = 2\times(-2) - 2^2 = -8 < 0$ and f_{xx} and f_{yy} have opposite signs, so this is a saddle point.





2.3.5 Plotting procedure

Examiners like asking you to identify and classify stationary points, and to sketch contours of the associated functions. The following steps might help you achieve this for a function of the form $z = f(x,y)$.

1 Identify and plot any easy contours, *e.g.* $f(x,y) = 0$.

2 Identify any symmetries in the problem.

a Axisymmetric problems can have at most one extremum and this will be at the origin. Locations where $\partial f/\partial r = 0$ for $r > 0$ will be circular ridges or valleys (or points of inflection). There will be no saddle points.

b If the function is odd in one direction, say about $y = 0$, then look for saddle points along $y = 0$. The locations of other stationary points will be arranged symmetrically about the line of symmetry, with corresponding extrema taking the opposite character on each side of the line.

c If the function is even in one direction, say about $x = 0$, then look for extrema along $x = 0$, or saddle points that have a contour perpendicular to $x = 0$.

3 Determine contours on which $\partial f/\partial x = 0$, and contours on which $\partial f/\partial y = 0$.

4 Find and classify stationary points.

5 Contours loop around extrema.

6 The contours cross at a saddle points.

7 There will be a pair of contours around any neighbouring extrema of the same type that touch/meet at a saddle point.

8 Contours from saddle points remain open to infinity, join neighbouring saddle points, or loop around extrema.

Exam questions are normally set with a particular method in mind. However, sometimes there are easier approaches if you have insight into the structure of the function. Provided you answer the question correctly, and justify your answer, then you will receive the credit.

2.3.6 Stationary point tripos examples

1996 Paper 1

Find the stationary values of

$$F(x,y) = 4x^2 + 4y^2 + x^4 - 6x^2y^2 + y^4$$

and classify them as maxima, minima or saddle points. Show the positions of the stationary points in the x–y plane and give a rough sketch of the contours of F.

Solution

Note the symmetry: we can interchange x and y, or replace x with $-x$, *etc*. We expect to see this symmetry show up in the resulting analysis.

The derivatives of

$$F(x,y) = 4x^2 + 4y^2 + x^4 - 6x^2y^2 + y^4$$

are

$$F_x = 8x + 4x^3 - 12xy^2 = 4x(2 + x^2 - 3y^2)$$

$$F_y = 8y - 12x^2y + 4y^3 = 4y(2 - 3x^2 + y^2)$$

$$F_{xx} = 8 + 12x^2 - 12y^2$$

$$F_{xy} = -24xy$$

$$F_{yy} = 8 - 12x^2 + 12y^2.$$

At the stationary points, we have $F_x = 0 = F_y$.

For $F_x = 0$¸we have either $x = 0$ or $2 + x^2 - 3y^2 = 0$.

At $x = 0$, $F_y = 0$ either when $y = 0$ or when $2 + y^2 = 0$. The second of these cannot happen, so we have a stationary point at $(x,y) = (0,0)$.

Considering $y = 0$ leads to $x = 0$ (the point we already have) or $2 + x^2 = 0$, which cannot occur.

We might also have stationary points when simultaneously

$$2 + x^2 - 3y^2 = 0$$

and $$2 - 3x^2 + y^2 = 0$$

$\Rightarrow$ $$8 - 8x^2 = 0$$

$\Rightarrow x = \pm 1$, and $y = \pm 1$.

Hence we have stationary points (x,y) at (0,0), (1,1), (1,–1), (–1,1) and (–1,– 1), with corresponding values of F of 0, 4, 4, 4 and 4.

At (0,0), $F_{xx} = 8 = F_{yy}$ and $F_{xx}F_{yy} - F_{xy}^2 = 8\times 8 - 0 > 0$, so this is a local minimum.

At (1,1), $F_{xx} = 8 = F_{yy}$ and $F_{xx}F_{yy} - F_{xy}^2 = 8\times 8 - (-24)^2 < 0$, so this is a saddle point.

Since the second derivatives only contain even powers of x and y, we get the same result for the other $(\pm 1,\pm 1)$ stationary points, so we have a local minimum and four saddle points.

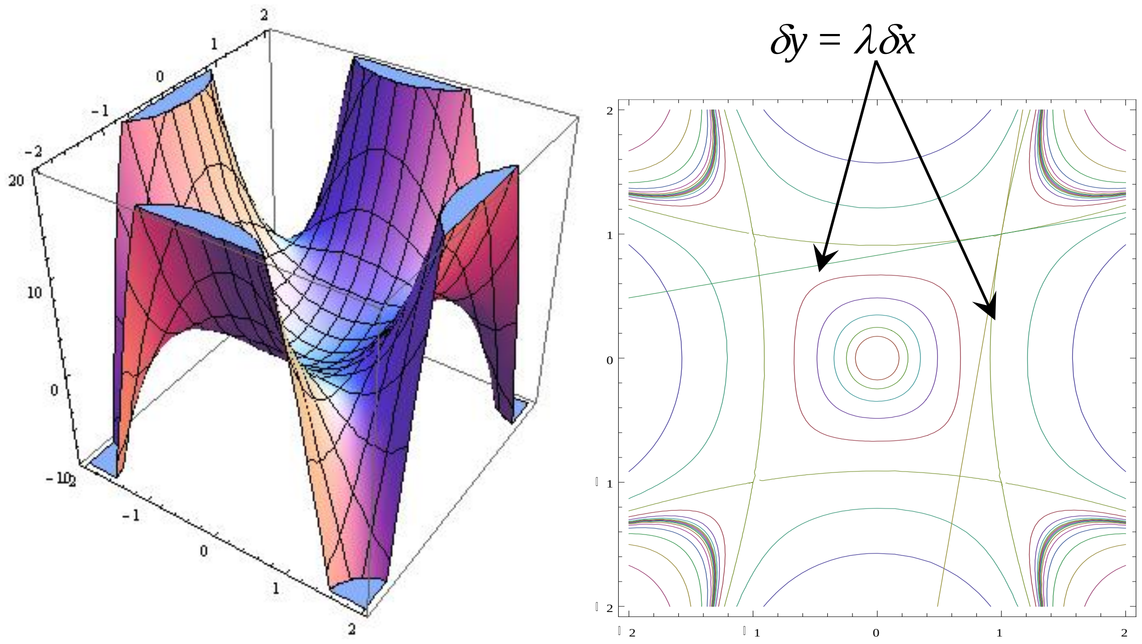
Although not essential, we could work out the orientation of the contours adjacent to the saddle points. At the (1,1) saddle point the function behaves locally as

$$F(1+\delta x,1+\delta y) = F(1,1) + \tfrac{1}{2}\delta x^2 F_{xx} + \delta x\,\delta y\,F_{xy} + \tfrac{1}{2}\delta y^2 F_{yy}$$
$$= 4 + 4\,\delta x^2 - 24\,\delta x\,\delta y + 4\delta y^2.$$

Taking $h = \lambda k$, then (locally) F is constant along lines given by

$$1 - 6\lambda + \lambda^2 = 0$$

$$\Rightarrow \qquad \left(\frac{dy}{dx}\right)_z = \lambda = 3 \pm \sqrt{9-1} = 3 \pm 2\sqrt{2}.$$



Contours shown at $f = 0, \pm 2^n$ for $n = -3, -2 \ldots 5$.

2010 Paper 2

(a) A function f of two variables x and y is defined as

$$f(x,y) = \exp\left[\frac{-1}{x^2+y^2}\right] + 3.$$

Find the position(s) of the stationary point(s) of f and determine the character (maximum, minimum or saddle) of each.

(b) A function g of two variables x and y is defined as follows:

$$g(x,y) = \sinh\left(y\sqrt{x^2+y^2} - 4y\right).$$

Sketch contours of g in the (x,y)-plane, making sure to indicate on the sketch the positions and character of all the stationary points, and making sure to label the heights of important contours or features.

Solution to (a)

This question was tough to do in a rigorous, mechanical manner.

$$f(x,y) = \exp\left[\frac{-1}{x^2+y^2}\right] + 3.$$

The axisymmetry of $f(x,y)$ means that it is obvious from the start that the only possible location for a stationary point is at the origin, and that it will be easiest to use the substitution $r^2 = x^2 + y^2$ so that

$$f(x,y) = f(r) = \exp\left(\frac{-1}{r^2}\right) + 3.$$

The examiners would have let you away with this statement, and asserting that inspection of f shows that since r increases away from the origin, then $1/r^2$ decreases so that $\exp(-1/r^2)$ increases, thus the origin must be a local minimum.

To be more rigorous, we should compute

$$\frac{df}{dr} = \frac{2\exp\left(-\frac{1}{r^2}\right)}{r^3}$$

and verify that this tends to zero as $r \to 0$. Doing so, however, is beyond the material you are expected to know in this course! The approach is stated here for completeness, **but you do not need to know how to do this**.

Using l'Hopital's rule
(You do not need to be able to do this)

Since both the numerator and denominator vanish at $r = 0$ we need to use l'Hopital's rule* to determine the behaviour. This is much easier if we use the substitution $\xi = 1/r$ and consider the limit as $\xi \to \infty$:

* l'Hopital's rule is beyond what you need to know for NST1A

$$\lim_{r\to 0}\frac{df}{dr}=\lim_{\xi\to\infty}\frac{2\xi^3}{\exp\left(\xi^2\right)}$$
$$=\lim_{\xi\to\infty}\frac{6\xi^2}{2\xi\exp\left(\xi^2\right)}=\lim_{\xi\to\infty}\frac{3\xi}{\exp\left(\xi^2\right)}$$
$$=\lim_{\xi\to\infty}\frac{3}{2\xi\exp\left(\xi^2\right)}$$
$$=0$$

Hence, there is a turning point at the origin. Computing

$$\frac{d^2 f}{dr^2}=\frac{d}{dr}\left(\frac{2\exp\left(-\frac{1}{r^2}\right)}{r^3}\right)=\frac{4-6r^2}{r^6}\exp\left(-\frac{1}{r^2}\right)$$

is not very helpful in determining the nature of the stationary point as the limit of this with $r\to 0$ vanishes and we would need to look at higher derivatives! However, inspection of f shows that since r increases away from the origin, then $1/r^2$ decreases so that $\exp(-1/r^2)$ increases, thus the origin must be a local minimum.

Solution to (b)

Again, this question was tough to approach in a mechanical manner.

$$g(x,y)=\sinh\left(y\sqrt{x^2+y^2}-4y\right)$$

Begin by evaluating the first derivatives of g and determining where these vanish:

$$\frac{\partial g}{\partial x}=\frac{d\sinh(\bullet)}{d\bullet}\frac{d\bullet}{dx}=\cosh\left(y\sqrt{x^2+y^2}-4y\right)\frac{xy}{\sqrt{x^2+y^2}}$$

which vanishes on $x = 0$ and on $y = 0$. (It also vanishes as $r^2 = x^2 + y^2 \to \infty$).

$$\frac{\partial g}{\partial y}=\frac{d\sinh(\bullet)}{d\bullet}\frac{d\bullet}{dy}$$
$$=\cosh\left(y\sqrt{x^2+y^2}-4y\right)\left(\frac{y^2}{\sqrt{x^2+y^2}}+\sqrt{x^2+y^2}-4\right)$$

On $x = 0$, we see that $\partial g/\partial y$ vanishes when

$$\left(\frac{y^2}{\sqrt{x^2+y^2}}+\sqrt{x^2+y^2}-4\right)=2|y|-4=0$$

$\Rightarrow$ $(x,y) = (0,-2)$ and $(0,2)$ are stationary points.

On $y = 0$ (where $\partial g/\partial x$ also vanishes), then $\partial g/\partial y$ vanishes when

$$\left(\frac{y^2}{\sqrt{x^2+y^2}}+\sqrt{x^2+y^2}-4\right)=|x|-4=0$$

$\Rightarrow$ $(x,y) = (-4,0)$ and $(4,0)$ are stationary points.

Computing the second derivatives:

$$\frac{\partial^2 g}{\partial x^2}=\frac{\partial}{\partial x}\left[\cosh\left(y\sqrt{x^2+y^2}-4y\right)\frac{xy}{\sqrt{x^2+y^2}}\right]$$

$$=\sinh\left(y\sqrt{x^2+y^2}-4y\right)\frac{x^2y^2}{x^2+y^2}$$

$$+\cosh\left(y\sqrt{x^2+y^2}-4y\right)\left(\frac{y\sqrt{x^2+y^2}-\frac{x^2y}{\sqrt{x^2+y^2}}}{x^2+y^2}\right)$$

$$=\sinh\left(y\sqrt{x^2+y^2}-4y\right)\frac{x^2y^2}{x^2+y^2}$$

$$+\cosh\left(y\sqrt{x^2+y^2}-4y\right)\left(\frac{y\left(x^2+y^2\right)-x^2y}{\left(x^2+y^2\right)^{3/2}}\right)$$

$$=\sinh\left(y\sqrt{x^2+y^2}-4y\right)\frac{x^2y^2}{x^2+y^2}$$

$$+\cosh\left(y\sqrt{x^2+y^2}-4y\right)\frac{y^3}{\left(x^2+y^2\right)^{3/2}}$$

$$\frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}\left[\cosh\left(y\sqrt{x^2+y^2}-4y\right)\left(\frac{y^2}{\sqrt{x^2+y^2}}+\sqrt{x^2+y^2}-4\right)\right]$$

$$= \sinh\left(y\sqrt{x^2+y^2}-4y\right)\left(\frac{y^2}{\sqrt{x^2+y^2}}+\sqrt{x^2+y^2}-4\right)^2$$

$$+\cosh\left(y\sqrt{x^2+y^2}-4y\right)\left(\frac{2y\sqrt{x^2+y^2}-\frac{y^3}{\sqrt{x^2+y^2}}}{x^2+y^2}+\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$= \sinh\left(y\sqrt{x^2+y^2}-4y\right)\left(\frac{y^2}{\sqrt{x^2+y^2}}+\sqrt{x^2+y^2}-4\right)^2$$

$$+\cosh\left(y\sqrt{x^2+y^2}-4y\right)\left(\frac{2y\left(x^2+y^2\right)-y^3+y\left(x^2+y^2\right)}{\left(x^2+y^2\right)^{3/2}}\right)$$

$$= \sinh\left(y\sqrt{x^2+y^2}-4y\right)\left(\frac{x^2+2y^2}{\sqrt{x^2+y^2}}-4\right)^2$$

$$+\cosh\left(y\sqrt{x^2+y^2}-4y\right)\left(\frac{3x^2y+2y^3}{\left(x^2+y^2\right)^{3/2}}\right)$$

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial}{\partial y}\left[\cosh\left(y\sqrt{x^2+y^2}-4y\right)\frac{xy}{\sqrt{x^2+y^2}}\right]$$

$$= \sinh\left(y\sqrt{x^2+y^2}-4y\right)\frac{xy}{\sqrt{x^2+y^2}}\left(\frac{y^2}{\sqrt{x^2+y^2}}+\sqrt{x^2+y^2}-4\right)$$

$$+\cosh\left(y\sqrt{x^2+y^2}-4y\right)\left[\frac{x\sqrt{x^2+y^2}-\frac{xy^2}{\sqrt{x^2+y^2}}}{x^2+y^2}\right]$$

$$= \sinh\left(y\sqrt{x^2+y^2}-4y\right)\frac{xy}{\sqrt{x^2+y^2}}\left(\frac{y^2}{\sqrt{x^2+y^2}}+\sqrt{x^2+y^2}-4\right)$$

$$+\cosh\left(y\sqrt{x^2+y^2}-4y\right)\left[\frac{x\left(x^2+y^2\right)-xy^2}{\left(x^2+y^2\right)^{3/2}}\right]$$

$$= \sinh\left(y\sqrt{x^2+y^2}-4y\right)\frac{xy}{\sqrt{x^2+y^2}}\left(\frac{x^2+2y^2}{\sqrt{x^2+y^2}}-4\right)$$

$$+\cosh\left(y\sqrt{x^2+y^2}-4y\right)\left[\frac{x^3}{\left(x^2+y^2\right)^{3/2}}\right]$$

At the stationary point $(x,y) = (0,2)$, we have

$$\frac{\partial^2 g}{\partial x^2} = \cosh(\bullet)\frac{y^3}{|y|^3} > 0$$

$$\frac{\partial^2 g}{\partial y^2} = \sinh\left(y\left(|y|-4\right)\right)(\bullet)^2 + \cosh(\bullet)\left(\frac{2y^3}{|y|^3}\right) > 0$$

$$\frac{\partial^2 g}{\partial x \partial y} = 0$$

hence this is a local minimum. At this point $g(0,2) = \sinh(-4)$.

Similarly, at $(x,y) = (0,-2)$ we have $g_{xx} < 0$, $g_{yy} < 0$ and $g_{xy} = 0$, so it is a local maximum with $g(0,2) = \sinh(4)$.

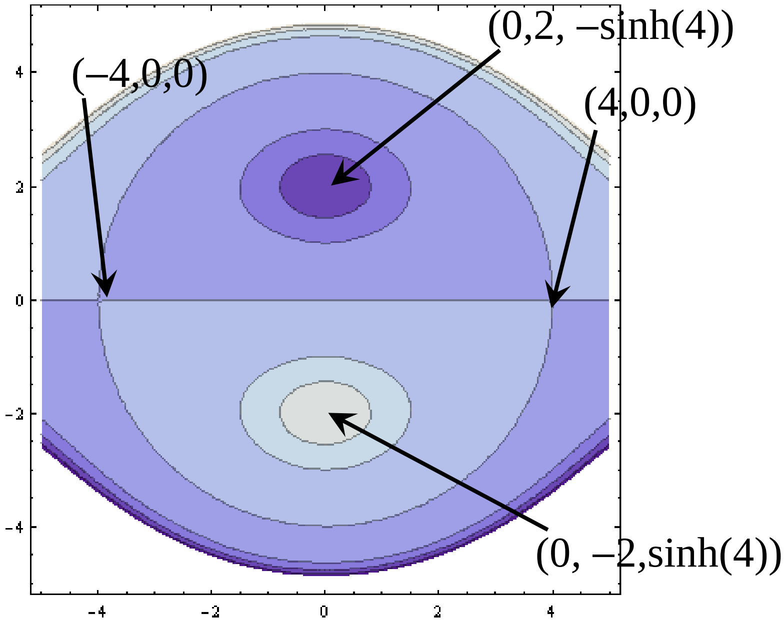
The other two stationary points $(x,y) = (\pm 4,0)$ give

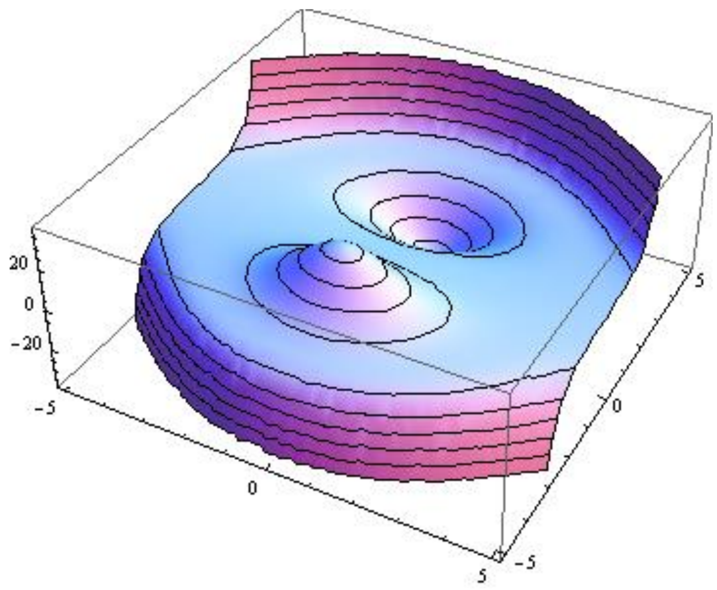
$$\frac{\partial^2 g}{\partial x^2} = 0 = \frac{\partial^2 g}{\partial y^2}$$

and
$$\frac{\partial^2 g}{\partial x \partial y} = \cosh(0)\left[\frac{x^3}{|x|^3}\right]$$

so that
$$g_{xx}g_{yy} - g_{xy}^2 = -x^2 < 0$$

and these two points are saddle points (with $g = 0$).





This question was harder and longer than most! It was the lowest scoring question on that paper. However, careful thought could have got us to the required answers much more quickly. We shall discuss, briefly, two other approaches.

Avoiding second derivatives

We could have determined this without computing the second derivatives simply by noting the symmetries:

1. $g(x,y) = \text{even}(x)\ \text{odd}(y)$ $\Rightarrow$ vanishes on $y = 0$
2. $g_x = \text{odd}(x)\ \text{odd}(y)$ $\Rightarrow$ vanishes on $x = 0$ and $y = 0$
3. $g_{xx} = \text{even}(x)\ \text{odd}(y)$ $\Rightarrow$ vanishes on $y = 0$
4. $g_y = \text{even}(x)\ \text{even}(y)$
5. $g_{yy} = \text{even}(x)\ \text{odd}(y)$ $\Rightarrow$ vanishes on $y = 0$
6. $g_{xy} = \text{odd}(x)\ \text{even}(y)$ $\Rightarrow$ vanishes on $x = 0$

and

7. at the origin $g_x = 0$,
8. at the origin $g_y = -4 < 0$.

That g_{xx} and g_{yy} vanish on $y = 0$ follows immediately from points (3) and (5), and hence $g_{xx}g_{yy} - g_{xy}^2 < 0$ gives saddle points at $(x,y) = (-4,0)$ and $(4,0)$.

Point (6) tells us that $g_{xy} = 0$ everywhere along $x = 0$, and so the nature of the extremum is determined solely by the signs of g_{xx} and g_{yy}.

Point (8) tells us that g_y must increase towards zero as we approach the extrema located on $x = 0$ by moving away from the origin (in either direction) along the y axis. Thus g_{yy} (an odd function of y) must be positive for $y > 0$ (and negative for $y < 0$).

Determining the sign of g_{xx} is less straight forward and cannot be done simply from the symmetries. However, inspection of $g(x,y) = \sinh\left(y\left(\sqrt{x^2+y^2}-4\right)\right)$ for fixed $y > 0$ shows us that increasing $|x|$ increases $\sqrt{x^2+y^2}-4$, the argument of sinh(..), and thus $g(x,y)$, so $g_{xx} > 0$ on $x = 0$ for positive y ($g_{xx} < 0$ for negative y).

Since $\partial^2 g/\partial x^2$ and $\partial^2 g/\partial y^2$ are both positive for $y > 0$, then the stationary point at (0,2) is a local minimum.

The best way (?)

We could have saved ourselves some work by rewriting

$$g(x,y)=\sinh\left(y\sqrt{x^2+y^2}-4y\right)=\sinh\left(y(r-4)\right)$$

and noting that the zero contour was a circle (of radius 4) and $y = 0$. Thus, since contours cross at $(x,y) = (\pm 4,0)$ those points must be saddles. [We should also note that for $r > 4$ the argument of sinh is monotonically increasing in magnitude, for any given y, and so there can be no stationary points beyond $r = 4$.]

The structure of the contours joining the two saddle points suggests there must be one local maximum and one local minimum, each lying within one of the semicircles created by the zero contour. The even x and odd y symmetry then tells us that these stationary points must lie on $x = 0$. By considering

$$g(0,y)=\sinh\left(y\left(|y|-4\right)\right)$$

it is obvious that $g(0,y)$ will be extremal when $|y| = 2$ since the argument to $\sinh(\cdot)$ is extremal at $y = \pm 2$ and the nature of the extrema.

2.4 Partial differential equations

2.4.1 What is a pde?

Ordinary differential equation

An *ordinary differential equation* relates the value of a function of a single variable, $f(x)$, say, through some combination of the function's (ordinary) derivatives, the function itself, and the independent variable. One way of expressing this general form of an ordinary differential equation of order n is as

$$F(x,f,f',f'',f''',\ldots,f^{(n)}) = 0,$$

where $f^{(n)}$ denotes the nth derivative of the function $f(x)$.

Alongside an ordinary differential equation of order n we need n additional pieces of information to determine the constants of integration associated with solving the equation. This additional information takes the form of boundary or initial conditions and may specify value(s) of the function and/or some of its derivatives.

Partial differential equation

As we have seen, if we have a function of more than one variable, we compute partial derivatives rather than ordinary derivatives. Each partial derivative has a direction associated with it, corresponding to one of the independent variables. A differential equation relating *partial derivatives* of a function of more than one variable is therefore termed a *partial differential equation* or *pde*.

We can write the general form of a partial differential equation of two variables as

$$F(x,y,f,f_x,f_y,f_{xx},f_{xy},f_{yy},f_{xxx},f_{xxy},f_{xyy},f_{yyy},\ldots) = 0.$$

As with an ordinary differential equation, we must provide boundary and/or initial conditions in order to move from a *general solution* to the specific solution for a particular problem. Typically, a solution is sought in some region D of the (x,y) plane, and information about f and/or its derivatives is given on certain parts (but not necessarily all) of the boundary ∂D of that region. For some types of problems, *initial values* may also be required for all points within D.

Within the above specification of a partial differential equation there are a huge number of possibilities. Many (most) cannot be solved analytically.

Here we shall restrict ourselves to examples of three classes of simple partial differential equations that occur widely in many branches of science. We shall show by substitution that certain functions and functional forms satisfy these equations. While there are more systematic ways of finding solutions of such equations that satisfy given boundary conditions, this is beyond what can fit in the present course.

2.4.2 The Poisson and Laplace equations

Poisson's equation (or *the Poisson equation*) in two dimensions is

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = s$$

where $s = s(x,y)$ is a known function.

There are a huge number of applications of this equation, ranging from the deformation of a membrane (a balloon) due to the pressure difference across it and steady temperature distributions in a heated plate, to fluid flow and electrostatics (where f is the potential, $\mathbf{E} = -\nabla f$ the electric field and $s = \varepsilon_0^{-1}$ multiplied by the charge density).

Laplace's equation (or *the Laplace equation*) is the special (homogeneous) case of Poisson's equation with $s = 0$. In two dimensions it is therefore simply

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Noting that ∇f is the vector (f_x,f_y) allows us to interpret ∇ as the *vector operator* $(\partial/\partial x,\partial/\partial y)$ that takes the scalar function f and returns the gradient vector.

We can therefore form the dot-product $\nabla\cdot\nabla$ which is then clearly the differential operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and thus we can write the Laplace equation as $\nabla\cdot\nabla f = 0$. The operator $\nabla\cdot\nabla$ is often abbreviated as ∇^2 (or sometimes Δ) and referred to as the *Laplacian* or *del-squared* or *nabla-squared*. When operating on a scalar function it returns a scalar that is the *curvature* of the function.

We can therefore write the Poisson equation as $\nabla^2 f = s$. This form generalises to functions of more than two variables. For example, in three dimensions

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Examples of the Laplace equation include:

- electrostatics in regions away from charges (*e.g.* the field outside a charged conducting surface; f would be the electrostatic potential)
- steady-state diffusion (*e.g.* steady-state diffusion of temperature with the temperature maintained at the boundaries; the steady-state distribution of a chemical resulting from sources and sinks at the boundaries of the region)
- the pressure field for a fluid flowing within a porous medium.

An important feature of both the Poisson and Laplace equations is that they are *linear*. It is therefore possible to add solutions of the *homogeneous equation* (*i.e.* the Laplace equation, in which all terms contain f) with a *particular integral* (*i.e.* a solution of the Poisson equation) to construct another solution.

In particular, if φ is a solution of the Laplace equation (*i.e.* $\nabla^2\varphi = 0$), and ψ is a solution of $\nabla^2\psi = s$, then $\beta = \psi + A\varphi$ also satisfies $\nabla^2\beta = s$.

Laplace equation

It is straight forward to write down some of the functions that satisfy the Laplace equation. In two dimensions,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

and polynomials such as x, y, xy, $x^2 - y^2$, $x^3 - 3xy^2$, $3x^2y - y^3$ can be shown to satisfy it. For example,

$f = x^3 - 3xy^2 \Rightarrow$ $\quad f_x = 3x^2 - 3y^2,$

$$f_y = -6xy$$

$$f_{xx} = 6x$$

$$f_{yy} = -6x$$

so $\quad \nabla^2 f = f_{xx} + f_{yy} = 6x - 6x = 0.$ ✓

A solution that is often useful in real problems is

$$f(x,y) = \ln(x^2 + y^2)$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} \qquad \frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2(x^2+y^2) - 2x(2x)}{(x^2+y^2)^2} = \frac{2(-x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{2(x^2+y^2) - 2y(2y)}{(x^2+y^2)^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{2((-x^2+y^2)+(x^2-y^2))}{(x^2+y^2)^2} = 0$$ ✓

Complex exponentials

Show that the functions $f = e^{ky} \sin kx$ and $g = e^{-px} \cos py$ are solutions of the Laplace equation, and that their sum $af + bg$ is also a solution for any pair of constants a, b.

Solution

The derivatives of $f = e^{ky} \sin kx$ are

$$f_x = k\,e^{ky} \cos kx \;\Rightarrow\; f_{xx} = -k^2 e^{ky} \sin kx$$

$$f_y = k e^{ky} \sin kx \;\Rightarrow\; f_{yy} = k^2 e^{ky} \sin kx$$

$$\Rightarrow \quad f_{xx} + f_{yy} = -k^2 e^{ky} \sin kx + k^2 e^{ky} \sin kx = 0$$ ✓

Similarly for $g = e^{-px}\cos py$:

$$g_x = -pe^{-px}\cos py \quad \Rightarrow \quad g_{xx} = p^2e^{-px}\cos py$$

$$g_y = -pe^{-px}\sin py \quad \Rightarrow \quad g_{xx} = -p^2e^{-px}\cos py$$

$$\Rightarrow \qquad g_{xx} + g_{yy} = p^2e^{-px}\cos py - p^2e^{-px}\cos py = 0 \qquad ✓$$

Noting that $f_{xx} = -k^2f$, $f_{yy} = k^2f$, $g_{xx} = p^2g$ and $g_{yy} = -p^2g$ then

$$\nabla^2(af + bg) = -ak^2f + ak^2f + bp^2g - bp^2g = 0.$$

for any constants a and b.

Poisson equation

Show that $h = a\sin x\sin y$ is a solution of $\nabla^2h = \sin x\sin y$ and determine a Show further that $h + bf$ is a solution of the same Poisson equation where $f = e^{ky}\sin kx$.

Solution

The derivatives of h are

$$h_x = a\cos x\sin y \quad \Rightarrow \quad h_{xx} = -a\sin x\sin y = -h$$

$$h_y = a\sin x\cos y \quad \Rightarrow \quad h_{yy} = -a\sin x\sin y = -h$$

hence $\nabla^2h = h_{xx} + h_{yy} = -2h = -2a\ \sin x\sin y = \sin x\sin y$

$$\Rightarrow \qquad -2a = 1$$

$$\Rightarrow \qquad a = -½ \qquad ✓$$

Now $\quad \nabla^2(h + bf) = \nabla^2h + \nabla^2(bf) = -2h + 0 = \sin x\sin y.$ ✓

Classification – You do not need to know this!

Classification of partial differential equations is not part of this course. The classifications and descriptions of them given here are not rigorous.

The Poisson and Laplace equations are examples of *elliptic equations*. In an elliptic equation, the value of the function at every point is affected by the value of the function at every other point. We need boundary conditions around the entire edge of the domain. Changing the boundary condition at one point will affect the entire solution (although the effect may decay very rapidly away from the point that was changed).

We do not (generally) find the Poisson or Laplace equation with time as one of the independent variables as it would raise serious issues about causality.

2.4.3 The diffusion equation*

The one-dimensional diffusion equation is

$$\frac{\partial f}{\partial t} = \kappa \frac{\partial^2 f}{\partial x^2},$$

where t is time, x is the spatial coordinate and κ is a constant representing diffusivity. In physically realistic problems κ is always positive.

Applications of this equation include

- molecular diffusion of a chemical substance through a medium (f would then represent the concentration of the chemical and κ its molecular diffusivity)
- diffusion of heat through a material (f would then be temperature and κ the thermal diffusivity)

What you don't need to know about the physics

A more accurate statement is that the *diffusive flux* of f is given by

$$Flux = \kappa \frac{\partial f}{\partial x}$$

* You do not need to understand the physics.

and by considering the conservation of f we can see that $\partial f/\partial t = (\partial/\partial x)Flux$, thus

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left(\kappa\frac{\partial f}{\partial x}\right).$$

However, in a broad range of cases we can treat the diffusivity κ as being independent of f and x, and so this simplifies to the standard diffusion equation

$$\frac{\partial f}{\partial t} = \kappa\frac{\partial^2 f}{\partial x^2}.$$

Thermal diffusion in an infinite bar

If we have a domain of infinite size, then there is no externally imposed length scale. This lack of externally imposed length scale leads to a significant simplification. We start by noting that the thermal diffusivity has dimensions of *length squared per time* (L^2T^{-1}) and that for the problem to make sense we need to find a length scale other than simply x.

In fact, the only way the parameters and variables (apart from x) can be put together to form a length scale is $(\kappa t)^{1/2}$. Since this is the only length scale on which the solution can vary, we seek a solution of the form

$$f(x,t) = F(x/(\kappa t)^{1/2}) = F(\eta),$$

where $\eta = x/(\kappa t)^{1/2}$ is dimensionless.*

Rewriting the diffusion equation in terms of η requires

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}F(\eta) = \frac{\partial \eta}{\partial t}F'(\eta) = -\frac{1}{2}\frac{x}{\kappa^{1/2}t^{3/2}}F'(\eta) = -\frac{1}{2t}\eta F'(\eta)$$

$$\frac{\partial f}{\partial x} = \frac{\partial \eta}{\partial x}F'(\eta) = \frac{1}{(\kappa t)^{1/2}}F'(\eta)$$

* This is about the point an exam question might start, namely to determine the form of $F(\eta)$ that satisfies the diffusion equation when $\eta = x/(\kappa t)^{1/2}$.

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{(\kappa t)^{1/2}} \frac{\partial}{\partial x} F'(\eta) = \frac{1}{\kappa t} F''(\eta)$$

$$\Rightarrow \qquad \frac{\partial f}{\partial t} - \kappa \frac{\partial^2 f}{\partial x^2} = -\frac{\eta}{2t} F' - \kappa \frac{1}{\kappa t} F'' = 0$$

$$\Rightarrow \qquad F'' + \frac{\eta}{2} F' = 0 .$$

We have thus reduced the original partial differential equation in x and t into an ordinary differential equation in η.

This second-order equation is first-order in F' and we can solve it by *separation of variables*:

$$\frac{F''}{F'} = -\frac{\eta}{2} \quad \Rightarrow \quad \ln F' = -\eta^2/4 + a \quad \Rightarrow \quad F' = \hat{A}\exp\left(-\eta^2/4\right)$$

where $\hat{A}$ is an arbitrary constant.

To complete the solution we integrate F' again to obtain

$$F(\eta) = \hat{A}\int_0^{\eta} e^{-s^2/4} ds + B = A\,\mathrm{erf}\left(\tfrac{1}{2}\eta\right) + B$$

where A and B are arbitrary constants and erf(..) is the *error function* defined as

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds .$$

Note that the error function arises in probability because of its relationship with the integral of the *normal distribution*.

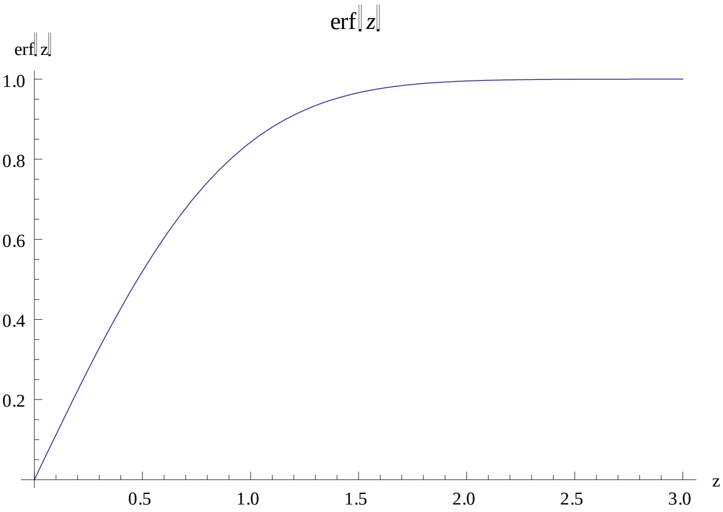


Figure 10: Error function.

This form of solution, where we can describe the behaviour in space and time using the dimensionless variable $\eta = x/\sqrt{\kappa t}$, is a simple form of what is referred to as a *similarity solution*; the variable η is the *similarity variable*. Such solutions arise surprisingly frequently in physical systems, and we frequently find that solutions originating from a different set of initial conditions are attracted towards such solutions.

Normally we would determine the constants A and B with reference to the initial conditions. For the moment, however, we take $A = 1$ and $B = 0$, say, and obtain a specific solution to the diffusion equation:

$$f_0(x,t) = \mathrm{erf}(\tfrac{1}{2}x/(\kappa t)^{1/2})$$

The error function has the property that $\mathrm{erf}(z) \to \pm 1$ as $z \to \pm\infty$, so $f_0(x,t) \to \pm 1$ as $x \to \pm\infty$ for $t > 0$.

For $t = 0$ the similarity variable $\eta = x/(\kappa t)^{1/2}$ is infinite for all x except $x = 0$. Thus, we find that $f_0(x,t{=}0)$ tends towards a *step function*.

Initial value problem

The function $f(x,t) = f_0(x,t)$ is the solution of the initial value problem

$$\frac{\partial f}{\partial t} = \kappa \frac{\partial^2 f}{\partial x^2} \text{ for } t > 0 \text{ with } f(x,0) = \begin{cases} -1 & x > 0 \\ 1 & x < 0 \end{cases}$$

The solution f_0 shows how diffusion smoothes out the initial 'discontinuity' in temperature, chemical concentration, or whatever else is being diffused.

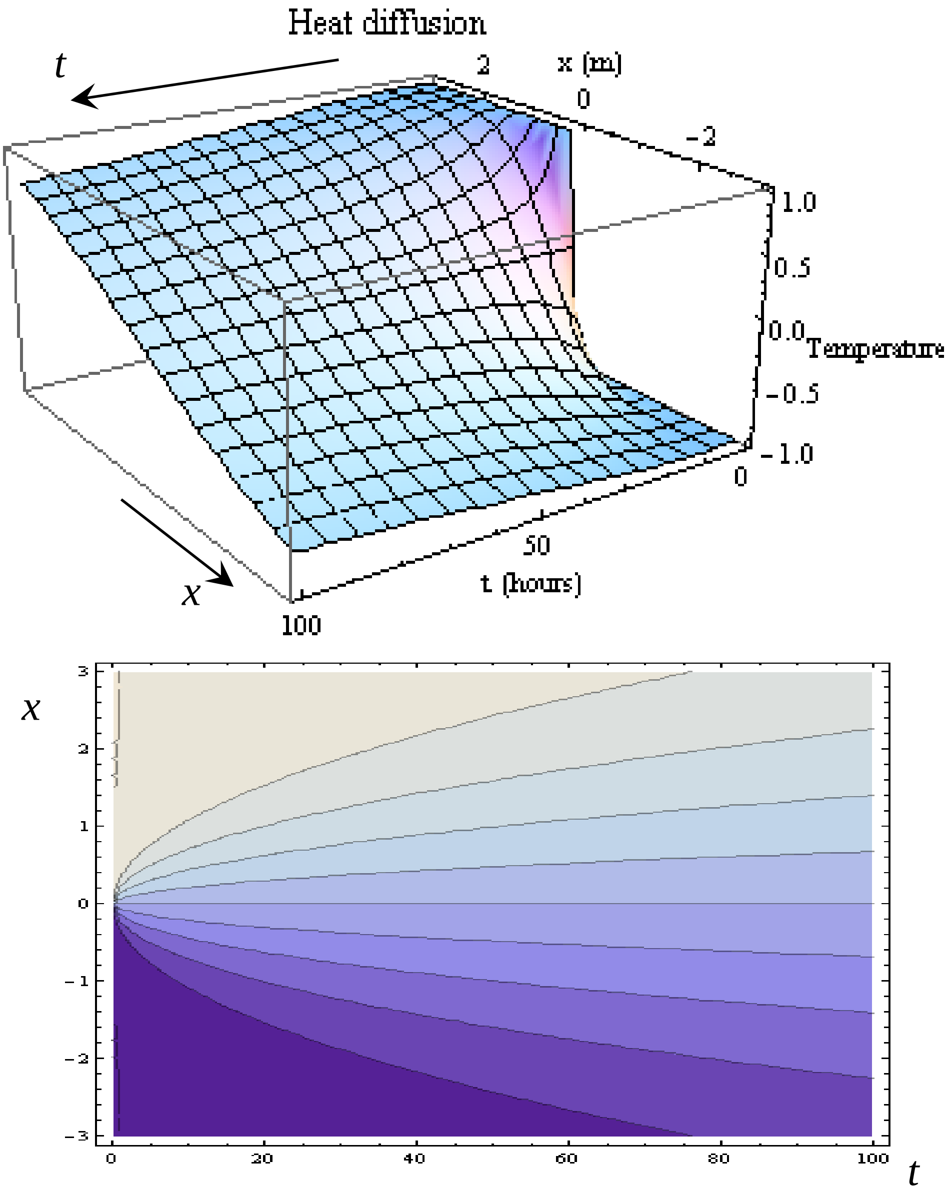


Figure 11: Thermal diffusion along a steel bar (time given in hours).

Thermal diffusivity – you do not need to know this!

The thermal diffusivity depends on the nature of the material the heat is diffusing in.

Material	Thermal diffusivity (m²/s) (m²/s)	Thermal conductivity W/m/K
Air (1 atm, 300 K)	2.2×10^{-5}	0.02
Expanded polystyrene	1.2×10^{-6}	0.03
Wood (Yellow Pine)	8.2×10^{-8}	0.12
Rubber	1.3×10^{-7}	0.13
Nylon	9×10^{-8}	0.25
Window glass	3.4×10^{-7}	0.96
Steel (1%)	1.1×10^{-5}	43
Copper	1.1×10^{-4}	401

Thermal diffusivity is a measure of how quickly the temperature changes, whereas thermal conductivity is a measure of the heat flux through the material. Note that expanded polystyrene has a very low thermal conductivity, but the thermal diffusivity is near the middle of the range! Conductivity (k) and diffusivity are related through the specific heat capacity (C_p) and density (ρ) by $\kappa = k/\rho C_p$.

Another solution

Note that if $f(x)$ is a solution of the diffusion equation, then so is $\partial f/\partial x$:

$$\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\right)-\kappa\frac{\partial^2}{\partial x^2}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial t}\right)-\frac{\partial}{\partial x}\left(\kappa\frac{\partial^2 f}{\partial x^2}\right)$$

$$=\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial t}-\kappa\frac{\partial^2 f}{\partial x^2}\right]=\frac{\partial}{\partial x}[0]=0$$

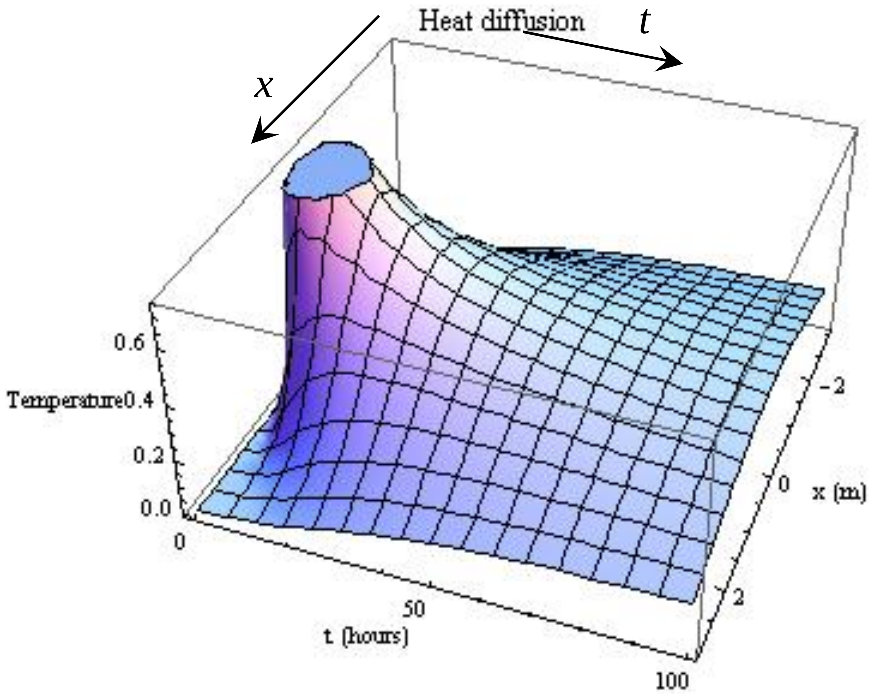
Thus another solution to the diffusion equation is

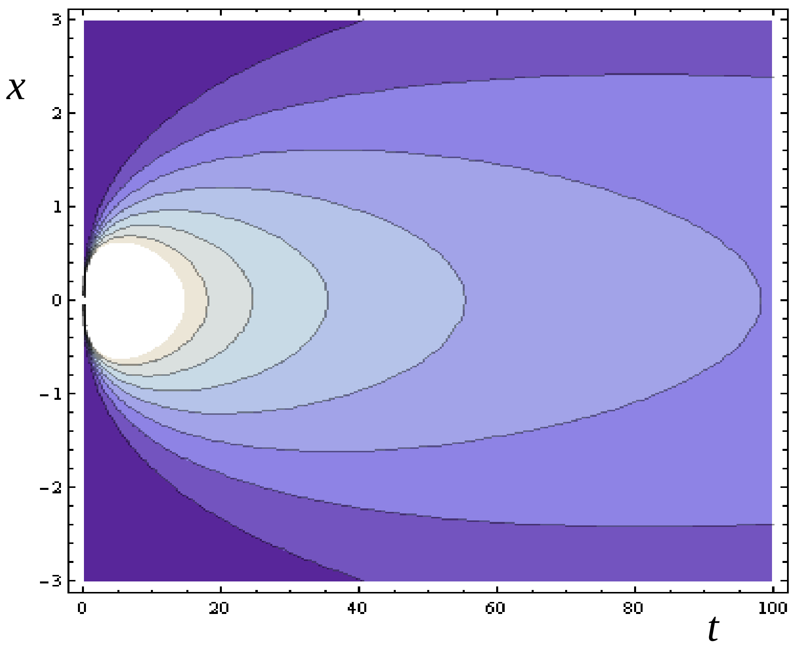
$$f_1(x,t)=\frac{\partial f_0}{\partial x}=\frac{\partial}{\partial x}\left[\operatorname{erf}\left(\frac{x}{(4\kappa t)^{1/2}}\right)\right]=\frac{1}{(\pi\kappa t)^{1/2}}e^{-x^2/4\kappa t}.$$

This solution has the property that $\int_{-\infty}^{\infty} f_1(x,t)\,dx = 2$ for all $t > 0$, so the total heat content does not change. It is clear that $f_1(x,t)$ is the solution of the initial value problem

$$\frac{\partial f}{\partial t} = \kappa \frac{\partial^2 f}{\partial x^2} \quad \text{for } t > 0 \text{ with } f(x,a) = \frac{1}{(\pi\kappa a)^{1/2}} e^{-x^2/4\kappa a}$$

As $a \to 0$, the initial conditions tend towards a *singular perturbation* at $x = 0$.





The solution f_1 shows how diffusion spreads out heat, chemicals, *etc.*, from a localised region where the temperature, concentration, *etc.*, is initially high and, at the same time, reduces the maximum temperature or concentration.

The solution f_1 does not depend solely on the similarity variable η, but η is still important. Although this is still a similarity solution, it is a more complex kind of similarity solution.

Exercise: Verify by partial differentiation that $f_0(x,t)$ and $f_1(x,t)$ are both solutions of the diffusion equation.

Although we have found two particular solutions of the diffusion equation, this does not provide us with a way of solving an initial value problem with arbitrary initial conditions.

Classification – You do not need to know this!

The diffusion equation of a *parabolic equations*. For a parabolic equation we need to have initial conditions. For a finite domain we also need to have boundary conditions. Changing the boundary condition at one point at $t = t_0$ will affect the entire solution at that instant and for later times. It will not affect the solution at earlier times ($t < t_0$).

2.4.4 The wave equation*

The one-dimensional wave equation

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}$$

describes the propagation of a wave with wave speed c. Here t is time and x is a spatial coordinate.

Examples:

- *small amplitude* waves on a stretched string (f would be the displacement)
- *small amplitude* waves on the surface of a river (f would be the change in the height of the surface)
- *small amplitude* plane electromagnetic waves (f would be a measure of the electric or magnetic field).

* You do not need to understand the physics.

If the waves are not small amplitude, then it is necessary to include additional terms in the equation, an area that is beyond the scope of this course.

When this equation is applied to a disturbance on a stretched string (*e.g.* a violin or guitar string), $f(x,t)$ is the lateral displacement of the string as a function of the distance along the string x and time t. The equation is therefore a statement about the sideways acceleration of the string ($\partial^2 f/\partial t^2$), a form of Newton's second law. The net force on a segment of string is related to the curvature ($\partial^2 f/\partial x^2$) of that portion of the string.

To solve the equation, we must also have a physically reasonable initial condition. This might take the form of an initial displacement $f(x,0)$ and velocity $\partial f/\partial t$ at $t = 0$; we could then follow the subsequent motion.

Infinite string

Consider a string of infinite length with initial conditions

$$f(x,0) = \begin{cases} x & 0 \le x < L \\ 2L - x & L \le x < 2L \\ 0 & elsewhere \end{cases}$$

$$\partial f/\partial t = 0 \text{ at } t = 0 \text{ for all } x.$$

To solve, we note that the linear wave equation has general solutions of a very simple form:

$$f(x,t) = F(x - ct) + G(x + ct),$$

where F and G are arbitrary functions (of a single variable). This general solution has derivatives

$$f_t = -cF'(x - ct) + cG'(x + ct)$$

$$f_{tt} = c^2F''(x - ct) + c^2G''(x + ct)$$

$$f_x = F'(x - ct) + G'(x + ct)$$

$$f_{xx} = F''(x - ct) + G''(x + ct),$$

which clearly satisfy $\dfrac{1}{c^2}\dfrac{\partial^2 f}{\partial t^2} = \dfrac{\partial^2 f}{\partial x^2}$.

This general solution is made up of two parts: $F(x - ct)$, a disturbance propagating to the right at speed c (in the positive x direction), and $G(x + ct)$, a disturbance propagating to the left at speed c (in the negative x direction).

The principal task is to determine $F(x)$ and $G(x)$ from our initial conditions at $t = 0$.

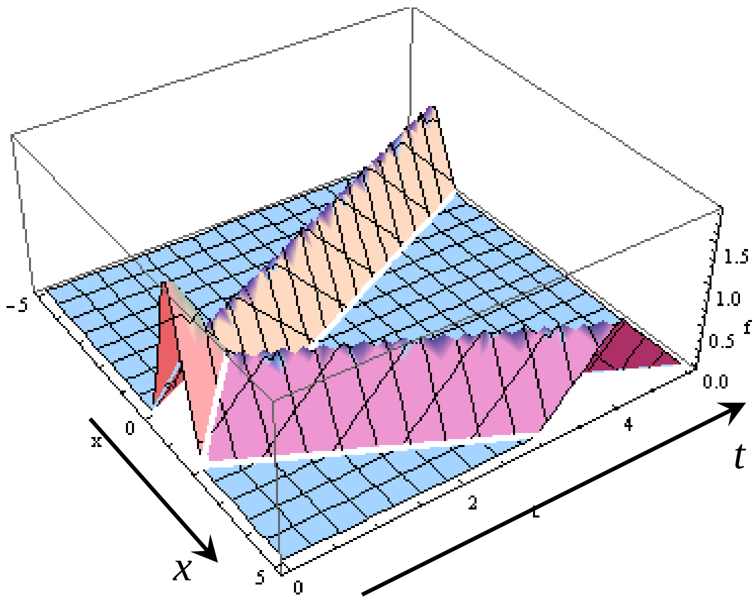
We have $\partial f/\partial t = -cF'(x) + cG'(x) = 0,$

which we can integrate immediately to determine that

$$F(x) = G(x) + const. \Rightarrow f(x,0) = 2F(x) + const.$$

Hence $F(x) = G(x) = \tfrac{1}{2} f(x,0).$

The solution is the sum of a wave travelling in the positive x direction and a wave travelling in the negative x direction.



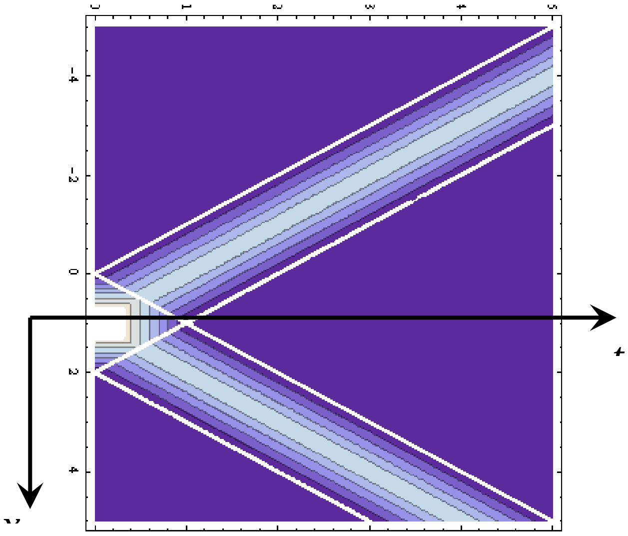


Figure 12: Solution of the 1D wave equation on an infinite string.

Finite length string

In a string of finite length, we need to consider the role of reflections from the end of the string. This leads to the idea of standing waves and resonance that give musical instruments their pitch and timbre, a subject beyond the scope of this course.

Classification – You do not need to know this!

The diffusion equation of a *hyperbolic equations*. For a hyperbolic equation we need to have initial conditions. We might also have boundary conditions. Information travels at a finite speed. Changing the boundary condition at one point at $t = t_0$ will take time to affect the solution at other points in the domain at later times. It will not affect the solution at earlier times ($t < t_0$).

2.4.5 PDE Tripos example

The lecture schedules do not require you to be able to *solve* partial differential equations, but rather to *verify the of solution to a partial differential equation by substitution*. Examiners may, however, call on you to use other bits of the course to achieve this.

2010 Paper 1

A function u of two variables x and t is defined by

$$u(x,t) = t^a y\left(b^a t^a x\right) \qquad (*)$$

where a and b are (non-zero) real constants, and y is an arbitrary function of a single variable.

(a) Write down expressions for the following derivatives in terms of y

(i) $$\frac{\partial u}{\partial t},$$

(ii) $$\frac{\partial^3 u}{\partial x^3}.$$

(b) Now assume that the (previously arbitrary) function $y(s)$ satisfies the homogeneous Airy equation

$$\frac{d^2 y}{ds^2} - sy = 0.$$

Use this fact to prove that the function $u(x,t)$ in (*) satisfies the reduced Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0$$

for a suitable choice of the (non-dimensional) real-valued constants a and b, whose values you should find.

Solution to (a)

The key step here is to recognise that we can write (*) as

$$u(x,t) = \hat{u}(s,t) = t^a y(s)$$

with $s = s(x,t) = b^a t^a x$. It is then easy to see that

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\left(t^a y(s)\right) \\
&= \frac{\partial\left(t^a\right)}{\partial t} y(s) + t^a \frac{dy}{ds}\frac{\partial s}{\partial t} \\
&= at^{a-1} y(s) + t^a \frac{dy}{ds} ab^a t^{a-1} x \\
&= at^{a-1} y + ab^a t^{2a-1} xy' \\
&= at^{a-1} y + at^{a-1} sy' \\
&= at^{a-1}\left(y + sy'\right)
\end{aligned}$$

and

$$\frac{\partial u}{\partial x} = t^a \frac{dy}{ds}\frac{\partial s}{\partial x} = b^a t^{2a} y'$$

$$\Rightarrow \qquad \frac{\partial^2 u}{\partial x^2} = b^a t^{2a} \frac{\partial y'}{\partial x} = b^a t^{2a} y'' \frac{\partial s}{\partial x} = b^{2a} t^{3a} y''$$

$$\Rightarrow \qquad \frac{\partial^3 u}{\partial x^3} = b^{3a} t^{4a} y'''$$

Solution to (b)

Now we are asked to assume that

$$y'' - sy = 0,$$

and consider reduced Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0$$

$$= at^{a-1}\left[y + sy'\right] + \left[b^{3a}t^{4a}y'''\right].$$

$$= t^{a-1}\left[ay + asy' + b^{3a}t^{3a+1}y'''\right]$$

Differentiating the Airy equation, $y'' - sy = 0$, we can write

$$y''' = sy' + y,$$

and use this to rewrite the KdV equation as

$$0 = \left[ay + asy' + b^{3a}t^{3a+1}y'''\right]$$

$$= \left[\left(a + b^{3a}t^{3a+1}\right)y + \left(as + sb^{3a}t^{3a+1}\right)y'\right]$$

For this to be true we require there to be no explicit t dependence within the square brackets, hence we must select $a = -1/3$. This gives

$$0 = \left[a\left(a + b^{3a}\right)y + s\left(a + b^{3a}\right)y'\right],$$

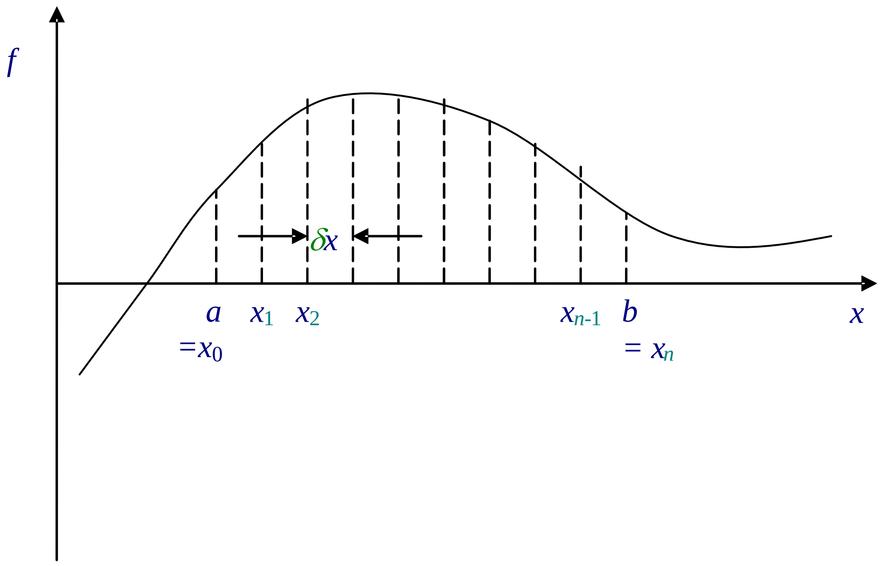
from which we require $a + b^{3a} = a + b^{-1} = 0$, hence $b = -1/a = 3$.

This was the worst scoring question of the paper and of the entire examination in 2010, yet it was relatively straight forward if one recognised how to approach it.

3. Multiple integration

3.1 Introduction

As we have seen for functions of a single variable, we can consider integration as the limit of a sum.



Let "dx_r" = $x_r - x_{r-1} = \delta x$, and $N\delta x = b - a$.

The area under the curve $f(x)$ over the range $a \le x \le b$ is given by the integral

$$I = \int_a^b f(x)\,dx .$$

This integral can be approximated by the sum

$$S_N = \sum_{i=1}^{N} f(x_i)\,\delta x ,$$

where $\delta x = (b - a)/N$ and we have divided the range $[a,b]$ into N subintervals $\xi_{i-1} \le x \le \xi_i$ with $\xi_i = a + i\delta x$. We can take each x_i anywhere in the corresponding subinterval, $\xi_{i-1} \le x_i \le \xi_i$.

As N becomes very large, the sum S_N tends towards the integral I,

$$I = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{i=1}^{N} f\left(x_i\right)\delta x$$

For a simple sum such as this, if we take x_i at a constant interval then we are in general best to select x_i in the middle of the subrange as this will produce the most accurate result. However, it is better again to have the x_i unevenly spaced, with their precise positioning chosen to maximise the accuracy. These ideas of *quadrature* are covered in the NST1B Mathematics course.

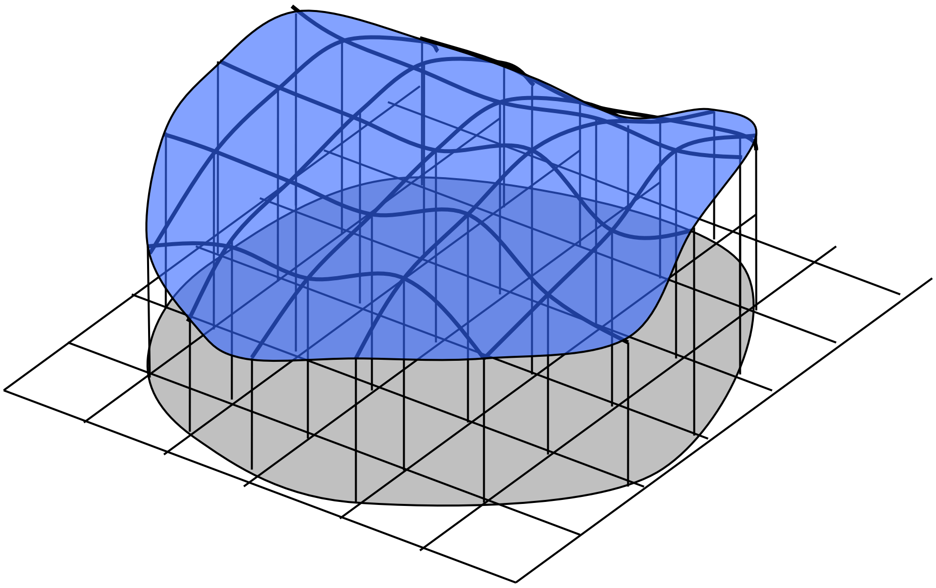
3.2 Double integrals

3.2.1 Definition

We frequently want to integrate functions of more than one variable. For example, if $h(x,y)$ is the height of a pile of grain in the region A, we may wish to know the volume V of grain.

As a sum

We could approximate this by dividing the region into P elements of area δA_i,



then the volume can be approximated by

$$V_P = \sum_{i=1}^{P} h\left(x_i, y_i\right)\delta A_i\,,$$

where the point (x_i,y_i) falls somewhere within the area element δA_i.

By analogy with single integrals in 1D, we could define this sum as the integral

$$V = \int\int_A h(x,y)\,dA.$$

However, to make progress, we need to rewrite the elemental area, and the area over which we are integrating or summing, in terms of the coordinates x and y.

Rectangular region

In the simplest case, we could consider the area A as the rectangular region $a \le x \le b$ and $c \le y \le d$.

We could then divide this region into M divisions of size $\delta x = (b - a)/M$ in the x direction, and N divisions of size $\delta y = (d - c)/N$ in the y direction. The area of each of the $P = M \times N$ elements is then $\delta A = \delta x\ \delta y$ and we can rewrite the sum as

$$V_{MN} = \sum_{i=1}^{P} h(x_i, y_i)\,\delta A_i = \sum_{i=1}^{M}\sum_{j=1}^{N} h(\hat{x}_i, \hat{y}_j)\,\delta y\,\delta x$$

where $\hat{x}_i$ fall somewhere between $a + (i-1)\delta x$ and $a + i\delta x$, and $\hat{y}_j$ falls somewhere between $b + (j-1)\delta y$ and $b + j\delta y$.

Note that the order of the two sums can be reversed:

$$V_{MN} = \sum_{i=1}^{M}\sum_{j=1}^{N} h(\hat{x}_i, \hat{y}_j)\,\delta y\,\delta x = \sum_{j=1}^{N}\sum_{i=1}^{M} h(\hat{x}_i, \hat{y}_j)\,\delta y\,\delta x.$$

The natural analogy of this sum is the *double integral*

$$V = \int_a^b \left[\int_c^d h(x,y)\,dy \right] dx.$$

We compute the *inner integral*, $\int_c^d h(x,y)\,dy$ first treating x as a constant parameter. The result of the inner integral is independent of y, but still depends on x. Hence the *outer integral* is identical to the normal integration procedure.

As with the order of summation, the order in which we integrate can be interchanged

$$V = \int_a^b \left[\int_c^d h(x,y)\,dy \right] dx = \int_c^d \left[\int_a^b h(x,y)\,dx \right] dy$$

so that the inner integral is computed over x while holding y constant.

Comparing

$$V = \int_a^b \left[\int_c^d h(x,y)\,dy \right] dx.$$

with

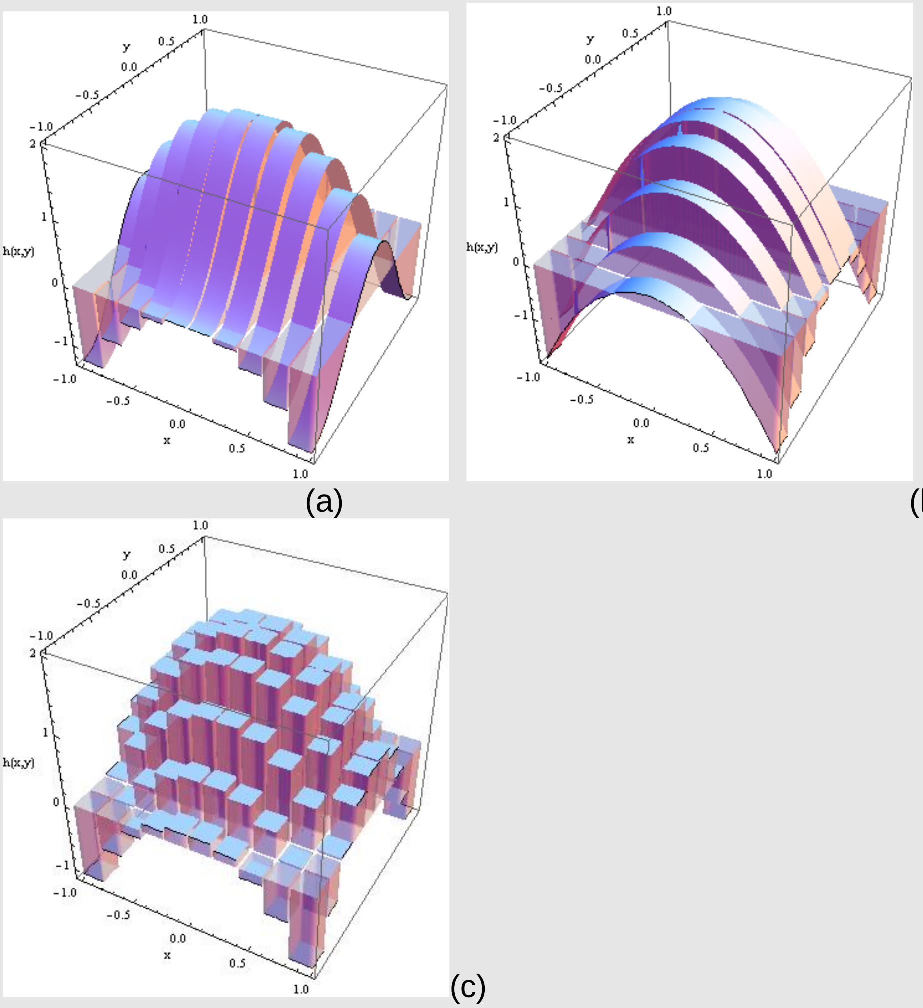
$$V = \int\int_A h(x,y)\,dA$$

shows clearly that $dA = dy\,dx$ and the area A corresponds to that given by the limits a, b, c and d.

Aside

This equivalence is analogous to expressing the double integral as the pair of sums of single integrals:

$$V \approx \sum_{i=1}^{M}\left[\delta x\int_{c}^{d} h\left(x_i, y\right) dy\right] \approx \sum_{j=1}^{N}\left[\delta y\int_{a}^{b} h\left(x, y_j\right) dx\right].$$



(a) (b) (c)

Figure 13: Approximations of double integral as sums of single integrals (a) & (b), or double sum of function values (c).

Other notations

We will often write the integral without the brackets as

$$V = \int_a^b \int_c^d h(x,y)\, dy\, dx\,.$$

There can be ambiguity as to which limit corresponds to which variable: most people/fields work outwards from the *integrand* (here $h(x,y)$) so that the dy corresponds to the *inner integral* with limits c and d, while the dx corresponds to the *outer integral* with limits a and b. However, sometimes this order is reversed. To avoid the ambiguity we sometimes write

$$V = \int_{x=a}^{x=b} \int_{y=c}^{y=d} h(x,y)\, dy\, dx \quad \text{or} \quad V = \int_{x=a}^{b} \int_{y=c}^{d} h(x,y)\, dy\, dx$$

to make it explicit which limits correspond to which variable.

In some fields (*e.g.* some branches of theoretical physics) it is common to place the increment immediately after the corresponding integral sign and write the integral as

$$V = \int_a^b \int_c^d dx\, dy\, h(x,y)$$

Sometimes this becomes

$$V = \int_a^b dx \int_c^d dy\, h(x,y).$$

which runs the risk of interpreting the equation as the product of two integral resulting in V as a function of x, namely

$$V(x) = \left[\int_a^b dx\right]\left[\int_c^d h(x,y)\, dy\right] = (b-a)\left[\int_c^d h(x,y)\, dy\right].$$

Obviously it is best to make your notation unambiguous, even if that means additional brackets. For this course, if multiple integration does not make it clear in other ways then you should interpret the integral in the following manner:

$$V = \int_a^b \int_c^d h(x,y)\, dy\, dx = \int_a^b \left[\int_c^d h(x,y)\, dy \right] dx .$$

> Examiners like to set questions using a range of notations to help reinforce that it is the meaning and techniques that is important, rather than the precise way an expression is written.

Non-rectangular region

We may readily extend these ideas if the region to be integrated over is not rectangular.

Suppose we want to integrate $h(x,y)$ over the region $y_0(x) \le y \le y_1(x)$ for $x_0 \le x \le x_1$. The area over which we are integrating is

$$A = \int_{x_0}^{x_1} y_1(x)\, dx - \int_{x_0}^{x_1} y_0(x)\, dx = \int_{x_0}^{x_1} \left[y_1(x) - y_0(x) \right] dx$$

and the double integral (the volume under $z = h(x,y)$) is

$$V = \int_{x_0}^{x_1} \left[\int_{y_0(x)}^{y_1(x)} h(x,y)\, dy \right] dx$$

We compute first the inner integral

$$I(x) \equiv H(x, y_1(x)) - H(x, y_2(x)) = \int_{y_0(x)}^{y_1(x)} h(x,y)\, dy ,$$

where $H(x,y)$ is the indefinite integral of $h(x,y)$ with respect to y. We have converted a function of both x and y into a function only of x. It is important to note here that the limits of this inner integral are themselves also functions of x.

Having computed the inner integral, we then compute the outer integral

$$V = \int_{x_0}^{x_1} I(x)\, dx .$$

Note that if $h(x,y) = 1$, then

$$V = \int_{x_0}^{x_1} \left[\int_{y_0(x)}^{y_1(x)} h(x,y)\, dy \right] dx = \int_{x_0}^{x_1} \left[\int_{y_0(x)}^{y_1(x)} 1\, dy \right] dx = \int_{x_0}^{x_1} [y]_{y_0(x)}^{y_1(x)}\, dx$$

$$= \int_{x_0}^{x_1} y_1(x) - y_0(x)\, dx$$

which is simply the area A over which we are integrating.

Changing the order of integration

We could, of course, have specified the region over which we are integrating by giving the boundary of it as $x_0(y)$ and $x_1(y)$ over the interval $y_0 \le y \le y_1$, expressing the integral as

$$V = \int_{y_0}^{y_1} \left[\int_{x_0(y)}^{x_1(y)} h(x,y)\, dx \right] dy .$$

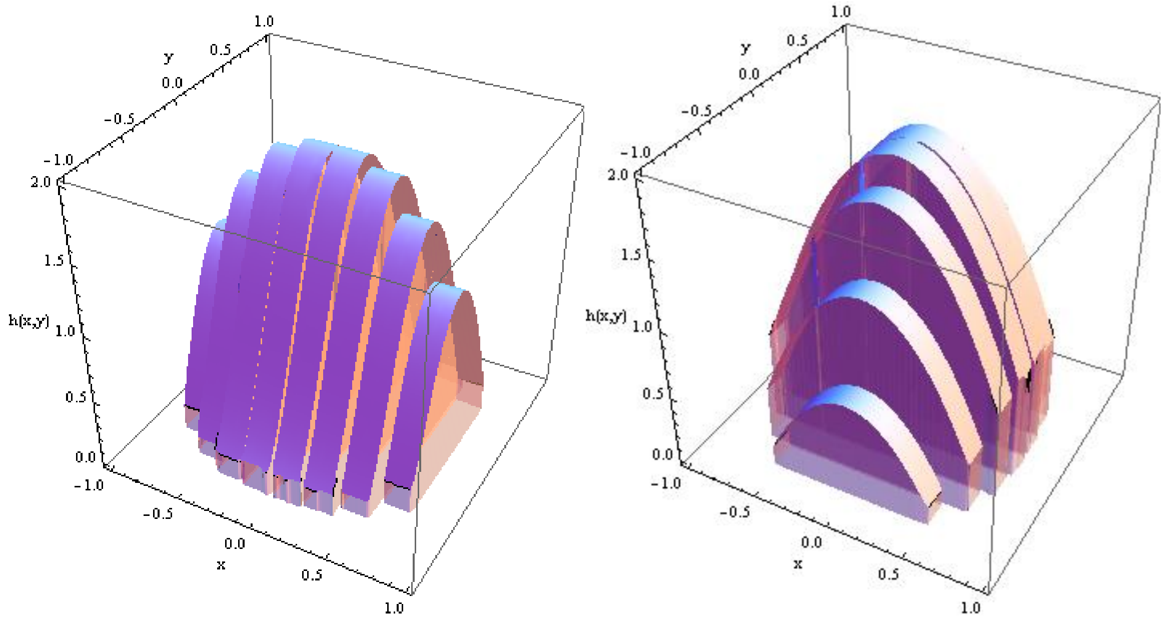


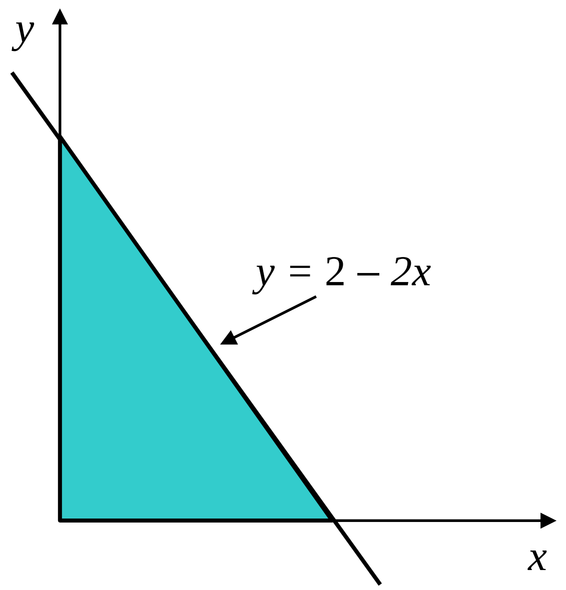
Figure 14: Approximation of double integral over nonrectangular region.

Our choice of the order of integration is often dictated by the need to express the limits of the inner integral in terms of the independent variable of the outer integral; it is not always possible to invert $x_0(y)$ and $x_1(y)$ into $y_0(x)$ and $y_1(x)$.

Integration over triangle

Evaluate the integral of xy over the triangle T enclosed by the lines $y = 0$, $x = 0$ and $y = 2 - 2x$:

$$I = \iint_T xy \, dA = \iint_T xy \, dx \, dy = \iint_T xy \, dy \, dx$$



We may evaluate this integral in two different ways.

Specifying the inner integral in terms of y:

$$I = \iint_T xy\, dA = \int_0^1 \left[\int_0^{2-2x} xy\, dy \right] dx$$

$$= \int_0^1 \left[\frac{1}{2} xy^2 \right]_{y=0}^{y=2(1-x)} dx = \int_0^1 2x(1-x)^2\, dx$$

$$= \int_0^1 2x - 4x^2 + 2x^3\, dx$$

$$= \left[x^2 - \frac{4}{3}x^3 + \frac{1}{2}x^4 \right]_0^1 = \frac{6-8+3}{6}$$

$$= \frac{1}{6}$$

Specifying the inner integral in terms of *x*:

$$y = 2(1-x) \rightarrow x = 1 - \tfrac{1}{2}y$$

$$I = \iint_T xy\, dA = \int_0^2 \left[\int_0^{1-\frac{1}{2}y} xy\, dx \right] dy$$

$$= \int_0^2 \left[\frac{1}{2} x^2 y \right]_{x=0}^{x=1-\frac{1}{2}y} dx = \int_0^2 \frac{1}{2}\left(1-\tfrac{1}{2}y\right)^2 y\, dx$$

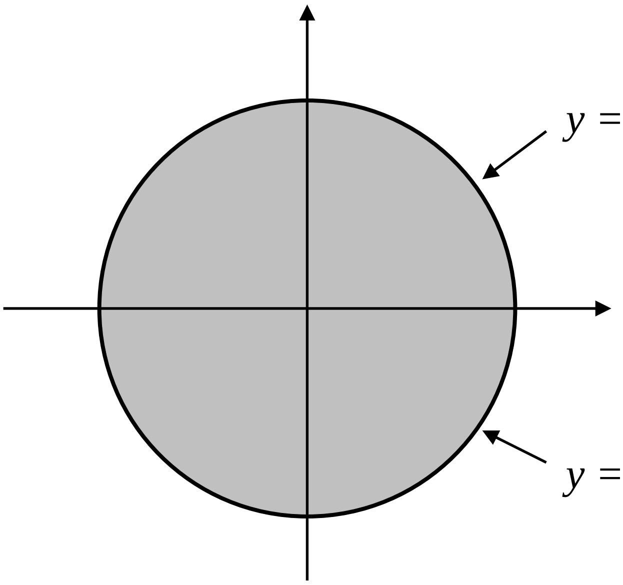
$$= \int_0^2 \tfrac{1}{2}y - \tfrac{1}{2}y^2 + \tfrac{1}{8}y^3\, dy$$

$$= \left[\frac{1}{4}y^2 - \frac{1}{6}y^3 + \frac{1}{32}y^4 \right]_0^2 = 1 - \frac{4}{3} + \frac{1}{2} = \frac{6-8+3}{6}$$

$$= \frac{1}{6}$$

Integral in a circle

By evaluating a double integral, find the area within the circle of radius a centred on the origin. (Call this region C.)



As we saw before, the area is the double integral of the function $f(x,y) = 1$ over the region (C). In terms of x, the upper and lower bounds on y are $y_1 = (a^2 - x^2)^{1/2}$ and $y_0 = -(a^2 - x^2)^{1/2}$.

Hence we are interested in the integral

$$I = \iint_C dA = \iint_C dx\,dy = \int_{-a}^{a}\left[\int_{-(a^2-x^2)^{1/2}}^{(a^2-x^2)^{1/2}} dy\right]dx$$

$$= \int_{-a}^{a} [y]_{-(a^2-x^2)^{1/2}}^{(a^2-x^2)^{1/2}}\,dx = \int_{-a}^{a} 2\left(a^2 - x^2\right)^{1/2} dx$$

Let $x = a\sin\theta \quad \Rightarrow \quad dx = a\cos\theta\; d\theta$

$$I = \int_{\theta=-\pi/2}^{\theta=\pi/2} 2a\left(1-\sin^2\theta\right)^{1/2} a\cos\theta\, d\theta$$

$$= 2a^2 \int_{\theta=-\pi/2}^{\theta=\pi/2} \cos^2\theta\, d\theta = \pi a^2$$

3.3 Integration in 2D polar coordinates

So far, we have specified the boundary of the region for the double integral in terms of Cartesian coordinates (x,y), and used these same coordinates to specify the function being integrated. However, this is can be awkward, as illustrated by the last example where we are specifying the region as a circle.

As an alternative, we could transform the function $f(x,y)$ to coordinates better suited to the region in which the integration is being performed. If the region is a circle, then one natural choice is two-dimensional polar coordinates (r,θ).

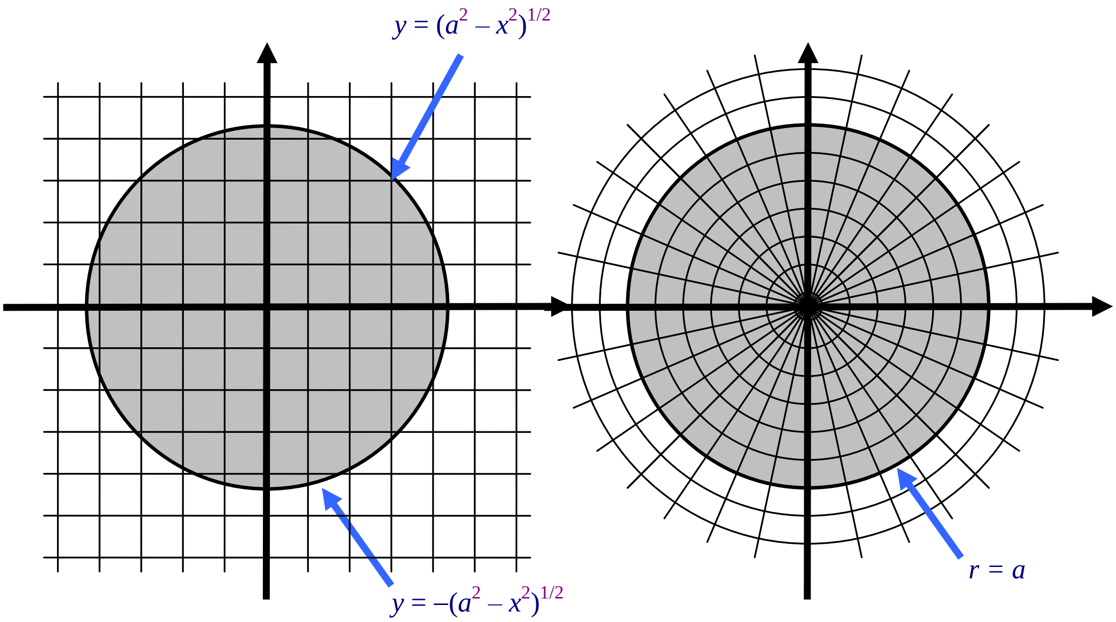
As we have seen when we considered the double integral as the limit of a sum, the small elemental areas used in the sum, δA_i were replaced by the product $\delta x\ \delta y$ and eventually by $dx\ dy$ when formulating the integral, *i.e.*

$$V=\sum_{i=1}^{P} h\left(x_i,y_i\right)\delta A_i=\sum_{i=1}^{M}\sum_{j=1}^{N} h\left(x_i,y_j\right)\delta y\ \delta x$$

and
$$V=\int\int_A h\left(x,y\right)dA=\int\int_A h\left(x,y\right)dx\ dy\,.$$

However, we do not have to have rectangular elements to represent dA or δA.

With two-dimensional polar coordinates we divide the integration region by many circles on which r is constant, and by many lines radiating from the origin on which θ is constant.



With two-dimensional polar coordinates, an element of area has a radial extent δr and azimuthal extent $\delta\theta$. Provided δr and $\delta\theta$ are small then the element is approximately rectangular of azimuthal width $r\delta\theta$ and an area

$$\delta A = r\ \delta r\ \delta\theta.$$

If we consider the volume under the function $f(r,\theta)$ for $r \le a$, then we can write this as

$$V = \sum_{i=1}^{P} f\left(r_i,\theta_i\right)\delta A_i = \sum_{i=1}^{M}\sum_{j=1}^{N} f\left(r_i,\theta_j\right) r_i \delta r \delta\theta = \sum_{j=1}^{N}\sum_{i=1}^{M} f\left(r_i,\theta_j\right) r_i \delta r \delta\theta$$

where $\delta r = a/M$ and $\delta\theta = 2\pi/N$.

The equivalent integral has $dA = r\,dr\,d\theta$ and is

$$V = \iint_C f\left(r,\theta\right) dA = \iint_C f\left(r,\theta\right) r dr d\theta$$

$$= \int_{-\pi}^{\pi}\left[\int_0^a f\left(r,\theta\right) r dr\right] d\theta = \int_0^a\left[\int_{-\pi}^{\pi} f\left(r,\theta\right) d\theta\right] r dr$$

Integrate over a circle

Integrate $f(x,y) = x^2 + y^2$ over a circle C of radius a centred on the origin:

$$I = \iint_C \left(x^2 + y^2\right) dx\,dy\,.$$

Now in polar coordinates, $r^2 = x^2 + y^2 \Rightarrow f(x,y) = r^2 = f(r,\theta)$ and

$$I = \iint_C \left(x^2 + y^2\right) dx\,dy = \iint_C \left(r^2\right) r\,dr\,d\theta$$

These two integrals are just different ways of writing the same thing. We have the freedom to choose either; obviously choosing the second will be easier!

$$I = \iint_C r^3\,dr\,d\theta = \int_{-\pi}^{\pi}\left[\int_0^a r^3\,dr\right]d\theta$$

$$= \int_{-\pi}^{\pi}\left[\tfrac{1}{4}r^4\right]_0^a d\theta = \frac{a^4}{4}\int_{-\pi}^{\pi} d\theta$$

$$= \frac{\pi a^4}{2}$$

Alternatively, we could have performed the integrations in the opposite order:

$$I = \iint_C r^3\,dr\,d\theta = \int_0^a\left[\int_{-\pi}^{\pi} r^3\,d\theta\right]dr = \int_0^a\left[r^3\int_{-\pi}^{\pi} d\theta\right]dr$$

$$= \int_0^a 2\pi r^3\,dr = \left[\frac{\pi}{2}r^4\right]_0^a$$

$$= \frac{\pi a^4}{2}$$

In the above we have made use of the fact that the integrand is independent of θ over the region (*i.e.* the integrand is circularly symmetric). If computing the volume as a sum, then we could have used many small circles of radius *r*, thickness *dr* and area $2\pi r dr$ and started from

$$I = \int\int_C r^2 \, dA = \int_0^a r^2 \, 2\pi r dr = 2\pi \int_0^a r^3 \, dr = \frac{\pi a^4}{2}.$$

Aside – You do not need to know this.

The integral computed here is the *second moment of area*. In particular, it is the integral of the area weighted by the square of the distance from the centroid. This quantity is closely related to the *moment of inertia* per unit mass of a circular disk of unit radius a. If rotating about the origin with an angular velocity ω then the kinetic energy of the disk will be $\frac{1}{2} I\omega^2$. [This expression should be compared with $\frac{1}{2} mu^2$, the kinetic energy of a mass m moving with velocity u.]

Separable integrand

If we can write $f(x,y) = g(x) \times h(y)$, and the limits on x and y are independent, then we can separate the double integral into the product of two single integrals:

$$I = \int_a^b \int_c^d f(x,y) \, dy \, dx = \int_a^b \left[\int_c^d g(x) h(y) \, dy \right] dx$$

$$= \int_a^b g(x) \left[\int_c^d h(y) \, dy \right] dx = \left[\int_a^b g(x) \, dx \right] \left[\int_c^d h(y) \, dy \right]$$

3.4 Triple integrals

We may extend the ideas of double integrals to *triple integrals* (and indeed even more nested integrals). Whereas the double integrals can be interpreted as determining a quantity integrated over an area, a triple integral can be viewed as representing a quantity integrated over a volume.

Instead of an elemental area dA that can be expressed as $dx\,dy$, we can consider an elemental volume dV that can be expressed as $dx\,dy\,dz$. In particular, we can write a triple integral as

$$I = \iiint_V f(x,y,z)\,dV = \iiint_V f(x,y,z)\,dx\,dy\,dz$$

corresponding to the limit of the sum

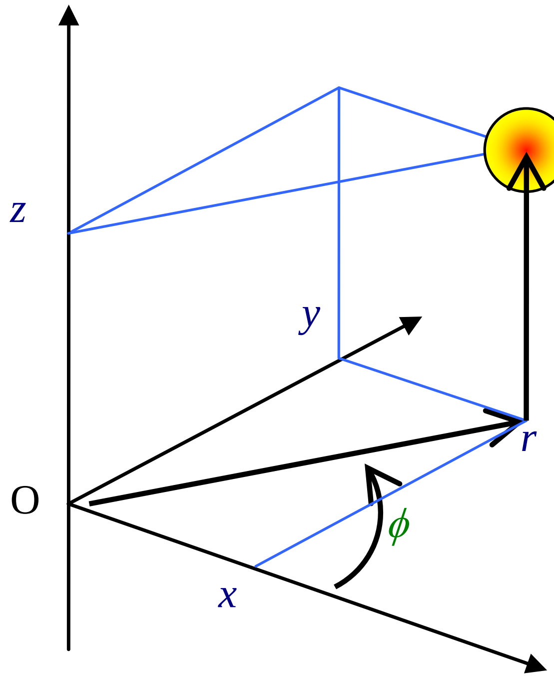
$$S_P = \sum_{i=1}^{P} f(x_i, y_i, z_i)\,\delta V_i$$

where the point (x_i,y_i,z_i) falls within the elemental volume δV_i.

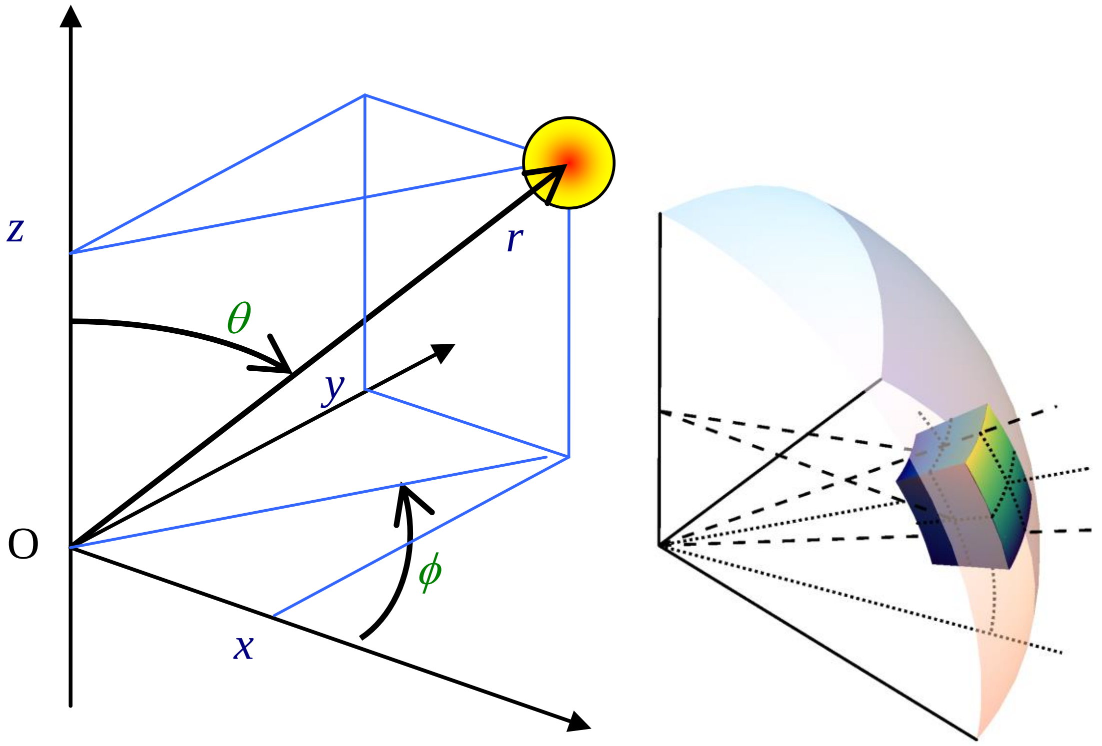
In the special case of $f(x,y,z) = 1$, then I is just the volume.

The elemental volume dV can be expressed in Cartesian coordinates as $dx\,dy\,dz$ (as above), or we can use some other coordinate system.

- In *cylindrical polar* coordinates, we replace (x,y,z) with (r,ϕ,z) and replace $dV = dx\,dy\,dz$ with $dV = r\,dr\,d\phi\,dz$. Note the similarity with two-dimensional polar coordinates. Also note that $r \geq 0$ and $-\pi < \phi \leq \pi$.



- In *spherical polar* coordinates (r,θ,ϕ) we use $dV = r^2 dr \sin\theta\, d\theta\, d\phi$. This is effectively the volume of a cuboid with sides dr, $r\,d\theta$ and $r\sin\theta\, d\phi$. It is important to remember that $r \geq 0$, $0 \leq \theta \leq \pi$ and $-\pi < \phi \leq \pi$ (or equivalently $0 < \phi \leq 2\pi$).



Conventions – You need to be aware of this

We need to be careful using spherical polar coordinates as there is no universally accepted convention for the order in which the coordinates are specified.

Here we use the order *radius* (r), *inclination angle* (θ), *azimuthal angle* (ϕ). This ordering is common practice in physics and is specified by ISO 31-11 (which defines mathematical signs and symbols for use in physical sciences and technology).

However, it is not uncommon to find the order of the two angles reversed!

The same ISO standard suggests that for cylindrical polar coordinates we should use ρ rather than r as the radius, giving (ρ,ϕ,z). Additionally, (r,θ,z) is very widely used due to the obvious connection with two-dimensional polar coordinates.

In an exam, read the question carefully to determine what each of the symbols means. If you introduce a different coordinate system in your answer, make it clear what means what.

We can handle triple integrals in a manner very similar to the way we treated double integrals. In particular:

1 Choose the coordinate system taking into account the shape of the volume V and the form of the integrand $f(x,y,z)$ to make the calculation as simple as possible.

2 Determine limits, *e.g.* $y_0(x) \le y \le y_1(x)$ and $x_0 \le x \le x_1$, or $0 \le r \le a$ and $0 \le \theta \le \pi/2$.

3 Rewrite the integrals (including integrand) in terms of the selected coordinate system.

4 Look to see if the integrand is separable and the limits independent of each other.

5 Decide on order in which to integrate

6 Integrate with respect to one variable at a time, working our way outward through all variables. Each integration eliminates one of the variables

Mass of a sphere

Calculate the mass of a sphere with uniform density ρ_0 and radius a.

$$M = \iiint_V \rho_0 \, dV .$$

It will obviously be easiest if we choose the origin at the centre of the sphere. Moreover, we expect spherical polar coordinates to be the best choice.

The sphere is therefore the region $0 \le r \le a$, $0 \le \theta \le \pi$ and $-\pi < \phi \le \pi$, and the volume element is $dV = r^2 \, dr \, d\phi \sin\theta \, d\theta$, so

$$M = \iiint_V \rho_0 \, dV = \int_0^{\pi} \left[\int_{-\pi}^{\pi} \left[\int_0^a \rho_0 \, r^2 dr \right] d\phi \right] \sin\theta \, d\theta$$

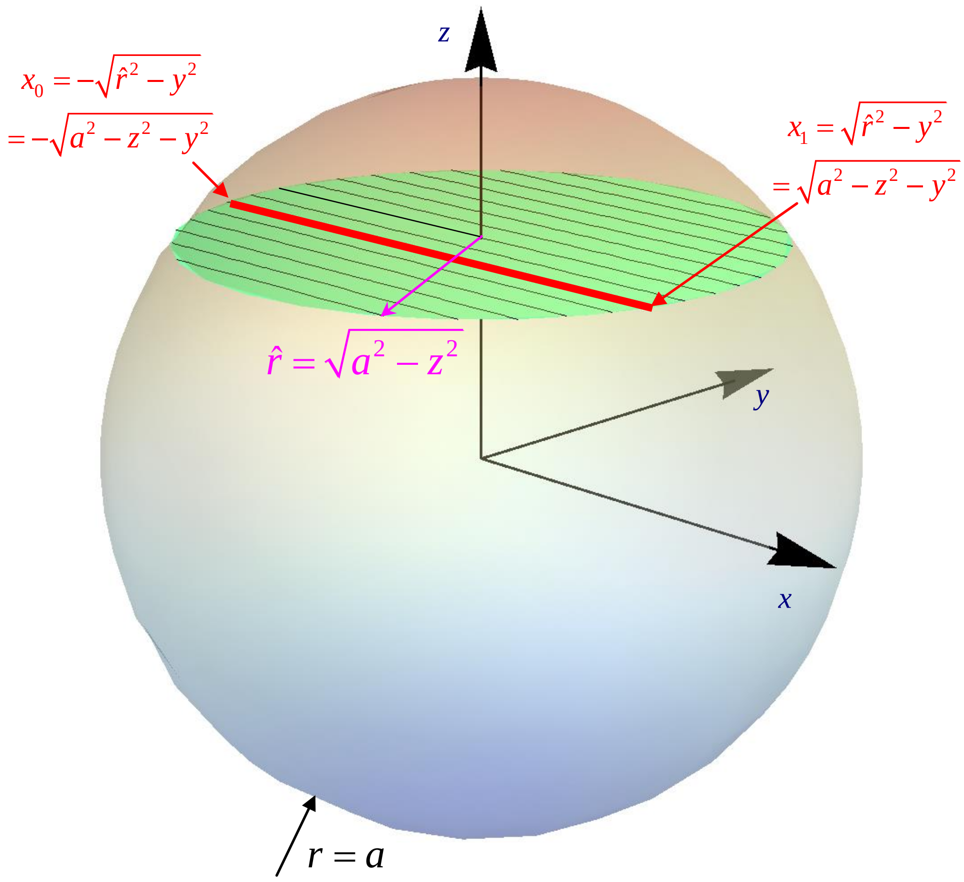
Since the limits are all independent, and the integrand can be factorised into parts containing no more than one of the independent variables, then we can rewrite this as

$$M = \left[\int_0^a \rho_0\, r^2 dr\right]\left[\int_{-\pi}^{\pi} d\phi\right]\left[\int_0^{\pi}\sin\theta\, d\theta\right]$$
$$= \left[\tfrac{1}{3}\rho_0 r^3\right]_0^a [\phi]_{-\pi}^{\pi} [-\cos\theta]_0^{\pi}$$
$$= \left[\tfrac{1}{3}\rho_0 a^3\right][2\pi][2]$$
$$= \frac{4}{3}\rho_0 \pi a^3$$

Not surprisingly, the mass is just the product of the density and the volume.

We could have evaluated the integral in Cartesian coordinates by noting that the sphere is the region

$$-(a^2 - y^2 - z^2)^{1/2} \le x \le (a^2 - y^2 - z^2)^{1/2},$$
$$-(a^2 - z^2)^{1/2} \le y \le (a^2 - z^2)^{1/2},$$
$$-a \le z \le a.$$



Thus

$$M = \iiint_V \rho_0 \, dV = \rho_0 \int_{-a}^{a} \left[\int_{-(a^2-z^2)^{1/2}}^{(a^2-z^2)^{1/2}} \left[\int_{-(a^2-y^2-z^2)^{1/2}}^{(a^2-y^2-z^2)^{1/2}} dx \right] dy \right] dz$$

$$= \rho_0 \int_{-a}^{a} \left[\int_{-(a^2-z^2)^{1/2}}^{(a^2-z^2)^{1/2}} [x]_{-(a^2-y^2-z^2)^{1/2}}^{(a^2-y^2-z^2)^{1/2}} \, dy \right] dz$$

$$= \rho_0 \int_{-a}^{a} \left[\int_{-(a^2-z^2)^{1/2}}^{(a^2-z^2)^{1/2}} 2\left(a^2 - y^2 - z^2\right)^{1/2} dy \right] dz$$

Making the substitution

$$y = (a^2-z^2)^{1/2} \sin\alpha \;\Rightarrow\; dy = (a^2-z^2)^{1/2} \cos\alpha \, d\alpha$$

$$M = \rho_0 \int_{-a}^{a} \left[\int_{-\pi/2}^{\pi/2} 2\left(a^2 - z^2\right)\cos^2\alpha \, d\alpha \right] dz = \rho_0 \int_{-a}^{a} \left[\pi\left(a^2 - z^2\right) \right] dz$$

$$= \rho_0 \pi \left[a^2 z - \tfrac{1}{3} z^3 \right]_{-a}^{a} = \frac{4}{3} \rho_0 \pi a^3$$

Mass of cylinder

Suppose the density of a cylinder of radius a and height $2h$, centred on the origin, varies as $\rho(x,y,z) = 1 + c(x^2 + z^2)$. Calculate the mass of the cylinder.

$$M = \iiint_V \rho \, dV = \iiint_V \rho(x,y,z) \, dx \, dy \, dz \, .$$

The circular symmetry of the domain suggest using cylindrical polar coordinates

$$M=\iiint_V \rho\, dV=\iiint_V 1+c\left(x^2+z^2\right) dx\, dy\, dz$$

$$=\int_{z=-h}^{z=h}\int_{\phi=-\pi}^{\phi=\pi}\int_{r=0}^{r=a}\left(1+cx^2+cz^2\right) r dr\, d\phi\, dz$$

$$=\int_{z=-h}^{z=h}\int_{\phi=-\pi}^{\phi=\pi}\int_{r=0}^{r=a}\left(1+cr^2\cos^2\phi+cz^2\right) r dr\, d\phi\, dz$$

$$=\int_{z=-h}^{z=h}\int_{\phi=-\pi}^{\phi=\pi}\left[\tfrac{1}{2}r^2+\tfrac{1}{4}cr^4\cos^2\phi+\tfrac{1}{2}cr^2z^2\right]_0^a d\phi\, dz$$

$$=\int_{z=-h}^{z=h}\int_{\phi=-\pi}^{\phi=\pi}\tfrac{1}{2}a^2+\tfrac{1}{4}ca^4\cos^2\phi+\tfrac{1}{2}ca^2z^2\; d\phi\, dz$$

$$=\int_{z=-h}^{z=h}\pi a^2+\tfrac{1}{4}\pi ca^4+\pi ca^2z^2\; dz$$

$$=\pi\left[a^2z+\tfrac{1}{4}ca^4z+\tfrac{1}{3}ca^2z^3\right]_{-h}^{h}$$

$$=2\pi a^2h\left(1+\tfrac{1}{4}ca^2+\tfrac{1}{3}ch^2\right)$$

In the special case of $c = 0$ this recovers the expected result of the volume of the cylinder $2\pi a^2 h$.

Note that the integrations above could have been performed in any order.

We could also have performed the integration in Cartesian coordinates, but it would have been more difficult.

Exercise: Verify the result by integrating in a different order.

3.5 The Gaussian integral

The integral of $\exp(-x^2)$ crops up frequently in science. We have seen it already (as the *error function* $\mathrm{erf}(\cdot)$) in the similarity solution of the one-dimensional diffusion equation (§2.4.3), and last term in statistics for the *normal distribution*.

Consider the integral

$$I_a = \int_{-a}^{a} e^{-x^2}\, dx .$$

Unfortunately, we cannot evaluate this analytically for general a. However, we can in the limit as $a \to \infty$, *i.e.*

$$I_\infty = \int_{-\infty}^{\infty} e^{-x^2}\, dx .$$

We note first that the integral I_∞ only makes sense if I_a remains *bounded* as $a \to \infty$. This means that I_a gets closer and closer to some limiting value (I_∞) as a gets larger.

Bounded integrals

An example of an integral that remains bounded is

$$J_a = \int_{-a}^{a} \frac{1}{1+x^2}\, dx = \left[\tan^{-1} x\right]_{-a}^{a} = 2\tan^{-1} a .$$

As a increases, J_a gets closer and closer to π. Therefore it makes sense to say that

$$J_\infty = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \pi .$$

For an integral to converge it requires that the integrand $f(x)$ tends to zero sufficiently quickly.

$\int_1^a \frac{1}{x}\, dx = \ln a$ is not bounded as $a \to \infty$, but $\int_1^a \frac{1}{x^2}\, dx = 1 - \frac{1}{a}$ is bounded as $a \to \infty$.

Rather than integrating I_a, we shall try to integrate I_a^2:

$$I_a^2 = \left[\int_{-a}^{a} e^{-x^2}\, dx\right]^2 = \left[\int_{-a}^{a} e^{-x^2}\, dx\right]\left[\int_{-a}^{a} e^{-y^2}\, dy\right]$$

Because the limits are constant, we can rewrite this product of two integrals as the double integral of a product

$$I_a^2 = \int_{x=-a}^{x=a}\int_{y=-a}^{y=a} e^{-\left(x^2+y^2\right)}\, dy\, dx$$

over the square domain $-a \le x \le a$, $-a \le y \le a$.

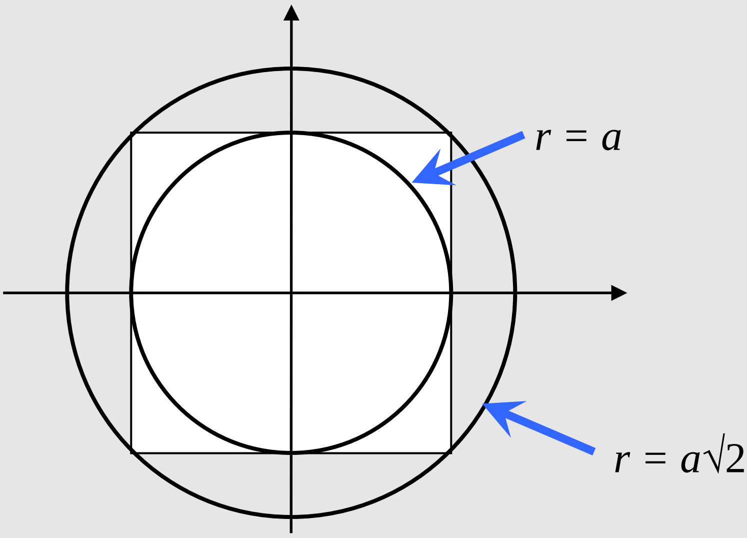
To determine I_∞ we note that the difference between an integral over the square domain with sides $2a$ and the integral over a circular domain of radius a will tend towards zero as $a \to \infty$. Thus since $r^2 = x^2 + y^2$ and $dxdy = r\, dr\, d\theta$ then

$$\begin{aligned} I_\infty^2 &= \int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty} e^{-\left(x^2+y^2\right)}\, dy\, dx \\ &= \int_{\theta=-\pi}^{\theta=\pi}\int_{r=0}^{r=\infty} e^{-r^2} r dr\, d\theta \\ &= \int_{\theta=-\pi}^{\theta=\pi} d\theta \int_{r=0}^{r=\infty} e^{-r^2} r dr \\ &= 2\pi \int_{r=0}^{r=\infty} e^{-r^2} r dr \\ &= \pi\left[-e^{-r^2}\right]_0^\infty = \pi \end{aligned}$$

Hence $I_\infty = \sqrt{\pi}$.

More rigorously – You do not need to know this

We could do this more rigorously by noting that since the integrand is always positive, the integral over the square of side $2a$ must fall between that over a circle of radius a (the *inscribed circle*) and that over a circle of radius $a\sqrt{2}$ (the *circumscribed circle*).



As both of these circular integrals tend towards the same limit, then I_a also tends towards this limit.

3.6 Extended integration examples

2004 Paper 1

A solid right circular cone C of height h and base radius a is bounded by the surfaces $r = a(1 - z/h)$ and $z = 0$ where r and z are cylindrical polar coordinates so that r is the distance from the axis of the cone. If the density ρ is given by

$$\rho(r) = \rho_0\left(1 + \frac{r}{a}\right),$$

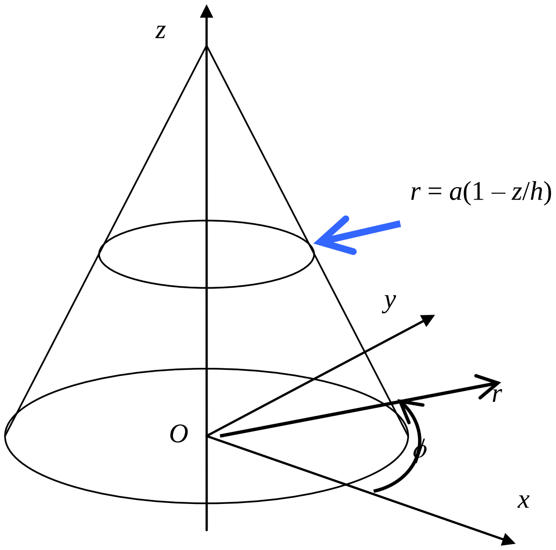
find the total mass $M = \int_C \rho\, dV$.

Find also the distance d of the centre of mass of the cone from its base, using the formula

$$Md = \int_C z\rho\, dV.$$

Solution

The cylindrical domain and axisymmetry of the density make it obvious that cylindrical polar coordinates are to be preferred.



We start by expressing the cone as $0 < r < a(1 - z/h)$, $-\pi < \phi \le \pi$, $0 \le z \le h$ and noting that the volume element is $dV = r\,dr\,d\phi\,dz$ so

$$M = \int_C \rho\, dV = \int_{z=0}^{z=h} \int_{\phi=-\pi}^{\phi=\pi} \int_{r=0}^{r=a(1-z/h)} \rho\, r dr\, d\phi\, dz.$$

Since the integrand is independent of ϕ, the limits for ϕ are constant and the limits for the other variables are independent of ϕ then we can take the $d\phi$ integral outside and write

$$\begin{aligned} M &= \int_{\phi=-\pi}^{\phi=\pi} d\phi \int_{z=0}^{z=h} \int_{r=0}^{r=a(1-z/h)} \rho\, r dr\, dz \\ &= 2\pi\rho_0 \int_{z=0}^{z=h} \int_{r=0}^{r=a(1-z/h)} \left(1+\frac{r}{a}\right) r dr\, dz \\ &= 2\pi\rho_0 \int_{z=0}^{z=h} \left[\frac{r^2}{2}+\frac{r^3}{3a}\right]_0^{a(1-z/h)} dz \\ &= 2\pi\rho_0 a^2 \int_{z=0}^{z=h} \left[\frac{1}{2}\left(1-\frac{z}{h}\right)^2+\frac{1}{3}\left(1-\frac{z}{h}\right)^3\right] dz \end{aligned}$$

At this point it is tempting to expand the integrand and integrate term by term:

$$\begin{aligned}M &= 2\pi\rho_0 a^2 \int_{z=0}^{z=h} \left[\frac{1}{2}\left(1 - 2\frac{z}{h} + \frac{z^2}{h^2}\right) + \frac{1}{3}\left(1 - 3\frac{z}{h} + 3\frac{z^2}{h^2} - \frac{z^3}{h^3}\right)\right] dz \\ &= \frac{1}{3}\pi\rho_0 a^2 \int_{z=0}^{z=h} \left[5 - 12\frac{z}{h} + 9\frac{z^2}{h^2} - 2\frac{z^3}{h^3}\right] dz \\ &= \frac{1}{3}\pi\rho_0 a^2 \left[5z - 6\frac{z^2}{h} + 3\frac{z^3}{h^2} - \frac{1}{2}\frac{z^4}{h^3}\right]_0^h \\ &= \frac{1}{3}\pi\rho_0 a^2 h \left[\frac{10 - 12 + 6 - 1}{2}\right] \\ &= \frac{1}{2}\pi\rho_0 a^2 h\end{aligned}$$

however this is messier than it need be, and it faster to integrate directly

$$\begin{aligned}M &= 2\pi\rho_0 a^2 \int_{z=0}^{z=h} \left[\frac{1}{2}\left(1 - \frac{z}{h}\right)^2 + \frac{1}{3}\left(1 - \frac{z}{h}\right)^3 \right] dz \\ &= 2\pi\rho_0 a^2 \left[-\frac{1}{6}h\left(1 - \frac{z}{h}\right)^3 - \frac{1}{12}h\left(1 - \frac{z}{h}\right)^4 \right]_0^h \\ &= 2\pi\rho_0 a^2 h \left[\frac{2+1}{12}\right] \\ &= \frac{1}{2}\pi\rho_0 a^2 h\end{aligned}$$

To determine Md (and hence d), we could simply substitute in and repeat the calculation. However, we can save some time if we note that

$$Md = \int_C z\rho\, dV = \int_C (z-h)\rho\, dV + h\int_C \rho\, dV = hM - h\int_C \left(1 - \frac{z}{h}\right)\rho\, dV$$

and so

$$Md = hM - h\int_C \left(1-\frac{z}{h}\right)\rho\, dV$$

$$= hM - h\int_{\phi=-\pi}^{\phi=\pi} d\phi \int_{z=0}^{z=h}\int_{r=0}^{r=a(1-z/h)} \left(1-\frac{z}{h}\right)\rho\, rdr\, dz$$

$$= hM - h\int_{\phi=-\pi}^{\phi=\pi} d\phi \int_{z=0}^{z=h} \left(1-\frac{z}{h}\right)\left[\int_{r=0}^{r=a(1-z/h)} \rho\, rdr\right] dz$$

We have already done the integral in square brackets when computing *M* so

$$Md = hM - 2\pi\rho_0 a^2 h\int_{z=0}^{z=h} \left(1-\frac{z}{h}\right)\left[\frac{1}{2}\left(1-\frac{z}{h}\right)^2 + \frac{1}{3}\left(1-\frac{z}{h}\right)^3\right] dz$$

$$= hM - 2\pi\rho_0 a^2 h\int_{z=0}^{z=h} \left[\frac{1}{2}\left(1-\frac{z}{h}\right)^3 + \frac{1}{3}\left(1-\frac{z}{h}\right)^4\right] dz$$

$$= hM - 2\pi\rho_0 a^2 h\left[-h\frac{1}{8}\left(1-\frac{z}{h}\right)^4 - h\frac{1}{15}\left(1-\frac{z}{h}\right)^5\right]_0^h$$

$$= hM - 2\pi\rho_0 a^2 h^2\left[\frac{15+8}{120}\right] = \pi\rho_0 a^2 h^2\left(\frac{1}{2}-\frac{23}{60}\right)$$

$$= \frac{7}{60}\pi\rho_0 a^2 h^2 = \frac{7}{30}Mh$$

This is a fairly tough question, especially if you expand the polynomials before integrating!

2005 Paper 1

(a) Evaluate

$$\int_1^2\int_1^2\int_1^2 \frac{dx\,dy\,dz}{x^2y^3z^4}.$$

(b) Evaluate using 2-D polar coordinates

$$\int_0^\infty dx\int_0^\infty dy\,\frac{yx^2}{x^2+y^2}e^{-\left(x^2+y^2\right)}.$$

Solution to (a)

$$\int_1^2\int_1^2\int_1^2 \frac{dx\,dy\,dz}{x^2y^3z^4}.$$

As the limits to the integral are all independent, and the integrand can be expressed as a product of functions of a single variable, then we may rewrite the integral as

$$\int_1^2\int_1^2\int_1^2 \frac{dx\,dy\,dz}{x^2y^3z^4} = \left[\int_1^2\frac{dx}{x^2}\right]\left[\int_1^2\frac{dy}{y^3}\right]\left[\int_1^2\frac{dz}{z^4}\right]$$

$$= \left[-\frac{1}{x}\right]_1^2\left[-\frac{1}{2}\frac{1}{y^2}\right]_1^2\left[-\frac{1}{3}\frac{1}{z^3}\right]_1^2$$

$$= -\left[\frac{1}{2}-1\right]\frac{1}{2}\left[\frac{1}{4}-1\right]\frac{1}{3}\left[\frac{1}{8}-1\right]$$

$$= \frac{1}{2}\frac{1}{2}\frac{3}{4}\frac{1}{3}\frac{7}{8} = \frac{7}{128}$$

Solution of (b)

We begin by rewriting the integral in the standard form we have used in this course:

$$I = \int_0^\infty dx \int_0^\infty dy \frac{yx^2}{x^2+y^2} e^{-(x^2+y^2)} = \int_0^\infty \int_0^\infty \frac{yx^2}{x^2+y^2} e^{-(x^2+y^2)}\, dy\, dx .$$

Now in 2D polar coordinates the first quadrant corresponds to $0 \le r < \infty$, $0 \le \theta \le \pi/2$, and we convert the increment $dA = dx\, dy$ into $dA = r\, dr\, d\theta$ so that

$$\begin{aligned} I &= \int_0^{\pi/2} \int_0^\infty \frac{y}{r}\left(\frac{x}{r}\right)^2 re^{-r^2}\, r\, dr\, d\theta \\ &= \int_0^{\pi/2} \int_0^\infty \sin\theta \cos^2\theta\, r^2 e^{-r^2}\, dr\, d\theta \end{aligned} .$$

Noting that the integrand is a product of a function of θ and a function of r, and the limits are independent, allows us to write

$$I = \left[\int_0^{\pi/2} \sin\theta \cos^2\theta\, d\theta\right]\left[\int_0^\infty r^2 e^{-r^2}\, dr\right]$$

The first integral is straight forward, while for the second we integrate by parts to get

$$\begin{aligned} I &= \left[-\frac{1}{3}\cos^3 x\right]_0^{\pi/2} \left(\left[-\frac{1}{2} re^{-r^2}\right]_0^\infty - \int_0^\infty -\frac{1}{2} e^{-r^2}\, dr\right) \\ &= \frac{1}{3}\int_0^\infty \frac{1}{2} e^{-r^2}\, dr = \frac{1}{6}\int_0^\infty e^{-r^2}\, dr \end{aligned}$$

This integral is very similar to the Gaussian integral we did in §3.5: it is just half of it. There we evaluated I_∞^2 in polar coordinates (to obtain the integrand $r \exp(-r^2)$) and found $I_\infty = \sqrt{\pi}$. Using this here we find

$$I = \frac{1}{6}\int_0^\infty e^{-r^2}\, dr = \frac{\sqrt{\pi}}{12}.$$

4. Scalar and vector fields

4.1 Introduction

Much of the contents of this chapter can be described as *vector calculus*. A loose definition of this is the differentiation and integration of vector quantities.

Notation

In this chapter we will be making extensive use of vectors. Within the typeset notes we will identify a vector variable using bold roman font, *e.g.* **u** or **q**. This notation is widely used in printed works. During the lectures vectors will be indicated using a ~ beneath the corresponding variable name, *e.g.* $\underset{\sim}{u}$ or $\underset{\sim}{q}$. The other commonly used notation (which we shall not use) puts an arrow above the variable name, *e.g.* $\vec{u}$ or $\vec{q}$.

We shall use more than one notation to represent the components of a vector. We represent the *unit vectors* in the *x*, *y* and *z* directions as **i**, **j** and **k**, respectively. Thus we can represent the vector **u** (in three dimensions) as

$$\mathbf{u} = \underset{\sim}{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = u_i .$$

Here the components are *a*, *b* and *c*, but sometimes we might use subscripts so that the components are u_1, u_2 and u_3.

Other common ways of representing the unit vectors include $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$, $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ and $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$. Here, we will only use vectors in Cartesian directions, although we may express the magnitudes of the components using other coordinate systems.

Scalar field

The idea of a *scalar field* can is closely related of a function of more than one variable, where at least some of the independent variables represent a spatial position. In particular, a scalar field $\sigma(x,y,z)$ assigns a scalar (a real number) to each point (x,y,z) in space.

Examples include

- temperature
- pressure
- chemical concentration
- density

Vector field

In contrast a *vector field* $\mathbf{F}(x,y,z)$ assigns a vector quantity (here indicated by bold roman typeface) to each point (x,y,z) in space. The vector quantity has a magnitude and direction; it is not simply an array of scalar quantities.

Examples include

- fluid velocity $\mathbf{U}(\mathbf{x})$
- magnetic field $\mathbf{B}(\mathbf{x})$
- electric field $\mathbf{E}(\mathbf{x})$.

Both the scalar field $\sigma(x,y,z)$ and vector field $\mathbf{F}(x,y,z)$ are functions of position. Here (x,y,z) represents a location in three-dimensional space relative to the origin. We can equivalently write this position as the vector $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are mutually orthogonal unit vectors, and write the scalar field as $\sigma(\mathbf{x})$ and vector field as $\mathbf{F}(\mathbf{x})$.

Number of spatial dimensions

These ideas do not rely on the number of spatial dimensions. In the above examples we assumed three dimensions, but there is a natural equivalence with two spatial dimensions or indeed with more than three dimensions. We shall look at examples in both two and three dimensions.

4.2 The gradient of a scalar field

We briefly introduced the idea of the *gradient* of a function in §2.2.1. In particular, we noted that the gradient of a function $f(x,y,z)$ was the vector (f_x,f_y,f_z). Here we apply these ideas to a scalar field.

We define the gradient of the scalar field $\sigma(\mathbf{x})$ as the vector field

$$\mathbf{v}(\mathbf{x}) = \text{grad}\,\sigma = \nabla\sigma = \frac{\partial\sigma}{\partial x}\mathbf{i} + \frac{\partial\sigma}{\partial y}\mathbf{j} + \frac{\partial\sigma}{\partial z}\mathbf{k}$$

The vector $\nabla\sigma$ defines the rate of change of $\sigma(\mathbf{x})$ with respect to position $\mathbf{x}$.

As we shall see later (in §4.5), a vector field defined in this way, *i.e.*, $\mathbf{v}(\mathbf{x}) = \nabla\sigma$, is referred to as a *conservative field* and has special properties.

If $\eta(\mathbf{x},t)$ is a scalar field that is a function of space $\mathbf{x}$ and time t, then we define the gradient (vector) field as

$$\mathbf{w}(\mathbf{x},t) = \frac{\partial\eta}{\partial x}\mathbf{i} + \frac{\partial\eta}{\partial y}\mathbf{j} + \frac{\partial\eta}{\partial z}\mathbf{k},$$

i.e. we only differentiate with respect to space $\mathbf{x}$ and *not* time t. For simplicity, most of the examples here will depend only on $\mathbf{x}$ and not on t.

Consider the change in σ between $\mathbf{x}$ and $\mathbf{x} + \delta\mathbf{x}$. Using a Taylor series expansion we may approximate this change as

$$\begin{aligned}\delta\sigma &\equiv \sigma(\mathbf{x}+\delta\mathbf{x})-\sigma(\mathbf{x})\\ &= \sigma(x+\delta x, y+\delta y, z+\delta z)-\sigma(x,y,z)\\ &\approx \frac{\partial\sigma}{\partial x}\delta x+\frac{\partial\sigma}{\partial y}\delta y+\frac{\partial\sigma}{\partial z}\delta z\\ &= \left(\frac{\partial\sigma}{\partial x}\mathbf{i}+\frac{\partial\sigma}{\partial y}\mathbf{j}+\frac{\partial\sigma}{\partial z}\mathbf{k}\right)\cdot(\delta x\mathbf{i}+\delta y\mathbf{j}+\delta z\mathbf{k})\\ &= \nabla\sigma\cdot\delta\mathbf{x}\end{aligned}$$

as $|\delta\mathbf{x}| \to 0$. This provides a very compact way of expressing the change in σ as $\mathbf{x}$ changes to $\mathbf{x} + \delta\mathbf{x}$ for any direction $\delta\mathbf{x}$.

Suppose $$\delta\mathbf{x} = \delta s\mathbf{q},$$

where $\mathbf{q}$ is a unit vector and $\delta s = |\delta\mathbf{x}|$ is the distance moved along $\mathbf{q}$. Hence

$$\delta\sigma = \delta s(\mathbf{q}\cdot\nabla\sigma).$$

Now consider $\sigma(\mathbf{x} + s\mathbf{q})$. We have already considered this as a Taylor series expansion in terms of $\delta\mathbf{x}$, but we can also consider it as a Taylor series expansion in terms of a single variable s. We may then estimate

$$\sigma(\mathbf{x}+\delta s\mathbf{q}) \approx \sigma(\mathbf{x})+\delta s\frac{d}{ds}\sigma(\mathbf{x}+s\mathbf{q})\Big|_{s=0}$$

hence $$\mathbf{q}\cdot\nabla\sigma = \frac{d}{ds}\sigma(\mathbf{x}+s\mathbf{q})\Big|_{s=0}$$

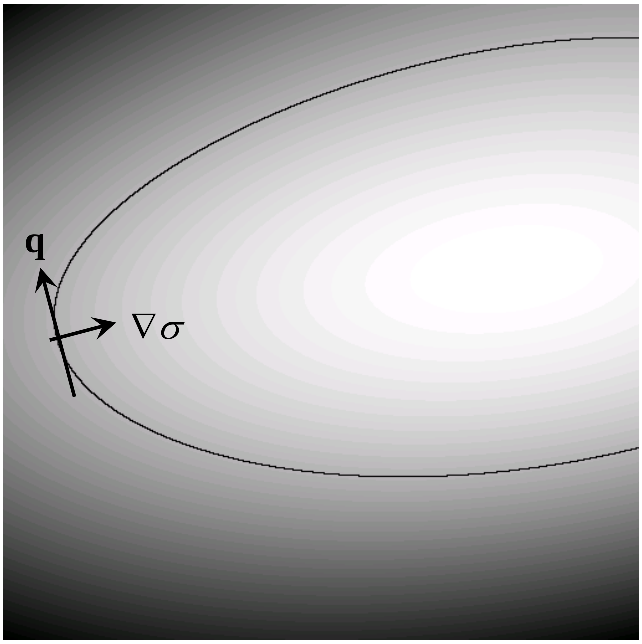
and $\mathbf{q}\cdot\nabla\sigma$ is the rate of change of σ in the direction $\mathbf{q}$. This is often referred to as the *directional derivative*.

Recall that the scalar product $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.

Now since **q** is a unit vector, $|\mathbf{q}| = 1$ and $\mathbf{q}\cdot\nabla\sigma = 1\times|\nabla\sigma|\times\cos\theta$, where θ is the angle between **q** and $\nabla\sigma$. Of course the rate of change is zero when $\theta = \frac{1}{2}\pi$, *i.e.* when **q** is perpendicular to $\nabla\sigma$.

Consider a curve (contour) in two dimensions (or a surface in three dimensions) on which σ is constant, *i.e.* $\{\mathbf{x}: \sigma(\mathbf{x}) = const\}$.

- If **q** is any vector tangential to the surface, then the rate of change of σ in the direction of **q** (evaluated at the point on the surface) is zero.
- Thus $\nabla\sigma$ is perpendicular to all **q** which are tangential to the surface
- Hence $\nabla\sigma$ is normal to surfaces of constant σ.



If **n** is the unit normal to a surface of constant σ, then $\mathbf{n} = \lambda\nabla\sigma$ for some scalar λ. However, as $|\mathbf{n}| = 1$, then $\lambda|\nabla\sigma| = 1$ so

$$\mathbf{n} = \frac{\nabla\sigma}{|\nabla\sigma|}$$

Example A

Consider $\sigma(x,y,z) = x^2 + y^2 + z^2$,

$$\Rightarrow \qquad \nabla\sigma = \frac{\partial\sigma}{\partial x}\mathbf{i} + \frac{\partial\sigma}{\partial y}\mathbf{j} + \frac{\partial\sigma}{\partial z}\mathbf{k} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = 2\mathbf{x}$$

Surfaces of constant σ are spheres centred on the origin. The unit normal vectors **n** to these surfaces are given by

$$\mathbf{n} = \frac{\nabla\sigma}{|\nabla\sigma|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{\left(4x^2 + 4y^2 + 4z^2\right)^{1/2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\left(x^2 + y^2 + z^2\right)^{1/2}} = \frac{\mathbf{x}}{|\mathbf{x}|}$$

Example B

Consider the surface defined by $z = \zeta(x,y)$. Calculate the normal to this surface at the point $(x,y,\zeta(x,y))$.

The surface $z = \zeta(x,y)$ is equivalent to

$$\sigma(x,y,z) = z - \zeta(x,y) = 0.$$

First calculate $\nabla\sigma$

$$\nabla\sigma = \nabla\left(z - \zeta\right) = -\frac{\partial\zeta}{\partial x}\mathbf{i} - \frac{\partial\zeta}{\partial y}\mathbf{j} + \mathbf{k}$$

the unit normal to $\sigma = const$ is then

$$\mathbf{n} = \frac{\nabla\sigma}{|\nabla\sigma|} = \frac{-\zeta_x\mathbf{i} - \zeta_y\mathbf{j} + \mathbf{k}}{\left(\zeta_x^2 + \zeta_y^2 + 1\right)^{1/2}}.$$

4.3 The gradient operator acting on vectors

Earlier, when we introduced ∇ (grad) of a scalar field *φ*, we noted that it can be considered as a *vector operator* acting on the scalar field. In particular, we can write

$$\nabla\varphi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\varphi = \begin{pmatrix} \dfrac{\partial\varphi}{\partial x} \\ \dfrac{\partial\varphi}{\partial y} \\ \dfrac{\partial\varphi}{\partial z} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{pmatrix}\varphi,$$

and so see that ∇ takes the form of a vector of differential operators. It is natural, therefore, to explore how ∇ can operate on vectors.

4.3.1 Divergence operator

The *divergence* of a vector field $\mathbf{U} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ is defined as the dot product (inner product or scalar product) of ∇ operating on the vector field **U**. In particular,

$$\begin{aligned}\mathrm{div}(\mathbf{U}) = \nabla\cdot\mathbf{U} &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\begin{pmatrix} u \\ v \\ w \end{pmatrix} \\ &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\end{aligned}$$

It is extremely important to note that although the inner product of two normal vectors is commutative, so that **a**·**b** = **b**·**a**, this is not true the divergence operator as (by convention) $u\partial/\partial x$ means something quite different from $\partial u/\partial x$.

By convention we interpret **U**·∇ as the scalar operator

$$\mathbf{U}\cdot\nabla = (u, v, w)\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$
$$= u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

We must be very careful when computing the divergence in non-Cartesian coordinates since the orientation of unit vectors are then themselves a function of space.

For example, we do not obtain the formula for $\nabla\cdot\mathbf{F}$ in cylindrical polar coordinates by simply replacing x by r and y by θ in the Cartesian expression if we want it to mean the same thing.

Sometimes, however, we may use polar coordinates to express the Cartesian vector components!

Differential operators in polar coordinates
You do not need to know this!

You do not need to know the form of the differential operators for non-Cartesian coordinates, but it is important to know that they are different and that swapping coordinate systems is not trivial.

Cylindrical polar coordinates

In cylindrical coordinates, (r,ϕ,z), the gradient of a scalar $\sigma(r,\phi,z)$ is

$$\nabla\sigma = \hat{\mathbf{r}}\frac{\partial\sigma}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial\sigma}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial\sigma}{\partial z},$$

where $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\phi}}$, $\hat{\mathbf{z}}$ are unit vectors in the corresponding directions.

The divergence of the vector field $\mathbf{u} = (u_r,u_\phi,u_z)$ is

$$\nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z}.$$

Spherical polar coordinates

In spherical coordinates (r,θ,ϕ),

$$\nabla\sigma = \hat{\mathbf{r}}\frac{\partial\sigma}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\sigma}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial\sigma}{\partial\phi}$$

$$\nabla\cdot\mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}$$

Example A

Calculate the divergence of the vector field $\mathbf{F} = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}$.

$$\begin{aligned}\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} &= \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)\\ &= a+b+c\end{aligned}$$

A special case is when $a = b = c = 1$, then $\mathbf{F} = \mathbf{x}$. Hence $\nabla\cdot\mathbf{x} = 3$.

4.3.2 Laplacian

Consider div grad $\varphi = \nabla\cdot\nabla\varphi$. This is equivalent to the divergence of a conservative vector field $\mathbf{F}$ defined as $\mathbf{F} = \nabla\varphi$. Hence

$$\begin{aligned}\operatorname{div}\operatorname{grad}\varphi = \nabla\cdot(\nabla\varphi) = \nabla\cdot\mathbf{F}\\ &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\\ &= \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}\\ &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi\\ &= \left[\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\right]\varphi\\ &= (\nabla\cdot\nabla)\varphi\end{aligned}$$

We encountered this operator, the *Laplacian operator*, in §2.4.2 in the Poisson and Laplace equations, where we wrote $\nabla\cdot\nabla\varphi$ as $\nabla^2\varphi$ (the notation $\Delta\varphi$ is also sometimes used).

A vector field $\mathbf{F}$ that satisfies $\nabla\cdot\mathbf{F} = 0$ is known as a *solenoidal* or an *incompressible* vector field. Hence, the scalar potential for a conservative solenoidal vector field must satisfy $\nabla^2\varphi = 0$.

Example B

Compute the Laplacian of $\varphi = \sin x \sin y \sin z$ by computing first the gradient then the divergence.

$$\nabla\varphi = (\cos x \sin y \sin z,\ \sin x \cos y \sin z,\ \sin x \sin y \cos z)$$

$$\begin{aligned}
\nabla\cdot(\nabla\varphi) &= \frac{\partial}{\partial x}(\cos x \sin y \sin z) + \frac{\partial}{\partial y}(\sin x \cos y \sin z) \\
&\quad + \frac{\partial}{\partial z}(\sin x \sin y \cos z) \\
&= -\sin x \sin y \sin z - \sin x \sin y \sin z - \sin x \sin y \sin z \\
&= -3 \sin x \sin y \sin z \\
&= -3\varphi
\end{aligned}$$

4.3.3 Curl

As we have seen, the divergence of a vector field is the inner or dot product of ∇ and the vector field. Thus the divergence of a vector field is a scalar.

The *curl* of a vector field is the *vector product* of ∇ and the vector field. In particular, if $\mathbf{U}(\mathbf{x})$ is a vector field, then

$$\begin{aligned}
\operatorname{curl}\mathbf{U} &= \nabla\times\mathbf{U} = \nabla\wedge\mathbf{U} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \\
&= \mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)
\end{aligned}$$

Remember that ∇ is a *vector operator* and so $\nabla\times\mathbf{U} \neq -\mathbf{U}\times\nabla$.

Example C

Calculate curl $\mathbf{F} = \nabla\times\mathbf{F}$ if $\mathbf{F} = \mathbf{a}\times\mathbf{x}$, where $\mathbf{a} = \mathbf{i}a + \mathbf{j}b + \mathbf{k}c$ is a constant vector.

Now $\mathbf{F} = \mathbf{a}\times\mathbf{x} = \mathbf{i}(bz - cy) + \mathbf{j}(cx - az) + \mathbf{k}(ay - bx)$

so

$$\nabla\times\mathbf{F} = \mathbf{i}\left(\frac{\partial}{\partial y}(ay-bx) - \frac{\partial}{\partial z}(cx-az)\right) + \mathbf{j}\left(\frac{\partial}{\partial z}(bz-cy) - \frac{\partial}{\partial x}(ay-bx)\right)$$
$$+\mathbf{k}\left(\frac{\partial}{\partial x}(cx-az) - \frac{\partial}{\partial y}(bz-cy)\right)$$
$$= 2a\mathbf{i} + 2b\mathbf{j} + 2c\mathbf{k}$$
$$= 2\mathbf{a}$$

A vector field $\mathbf{F}$ that satisfies $\nabla\times\mathbf{F} = 0$ is often referred to as an *irrotational* field.

4.3.4 curl grad and div curl

If $\mathbf{F} = \nabla\varphi = \mathbf{i}\,\partial\varphi/\partial x + \mathbf{j}\,\partial\varphi/\partial y + \mathbf{k}\,\partial\varphi/\partial z$, then

$$\nabla\times\mathbf{F} = \mathbf{0}$$

(a zero vector). Similarly,

$$\nabla\cdot(\nabla\times\mathbf{F}) = 0.$$

As we shall see later in §4.5, this is one way to test if a given vector field is *conservative* (*i.e.*, it can be expressed as $\mathbf{F} = \nabla\varphi$).

Exercise: prove these two identities.

Fundamental theorem – you do not need to know this

The fundamental theorem of vector calculus is that we can write *any* vector field **F** as the sum of a solenoidal and an irrotational field, thus we can write

$$\mathbf{F} = \nabla\varphi + \nabla\times\boldsymbol{\psi}\,.$$

The divergence of **F** then gives

$$\begin{aligned}\nabla\cdot\mathbf{F} &= \nabla\cdot(\nabla\varphi) + \nabla\cdot(\nabla\times\boldsymbol{\psi}) \\ &= \nabla^2\varphi\end{aligned}$$

and the curl of **F** gives

$$\begin{aligned}\nabla\times\mathbf{F} &= \nabla\times(\nabla\varphi) + \nabla\times(\nabla\times\boldsymbol{\psi}) \\ &= \nabla\times(\nabla\times\boldsymbol{\psi})\end{aligned}$$

The scalar field φ is known as the *scalar potential*, while the vector field $\boldsymbol{\psi}$ is the *vector potential*.

4.4 Line integrals

Previously, we have considered integrals of functions of one independent variable, and double and triple integrals of functions of more than one independent variable. Effectively, with one independent variable we integrate along a line, with two independent variables we integrate over an area, and with three we integrate over a volume.

Sometimes, however, when we have more than one independent variable we might still want to integrate along a line rather than over an area or throughout a volume. If the line corresponds to all but one of the variables being held constant, then it is obvious how to proceed. For example, if we want to evaluate $f(x,y)$ along the line $y = a$, then we only have to consider

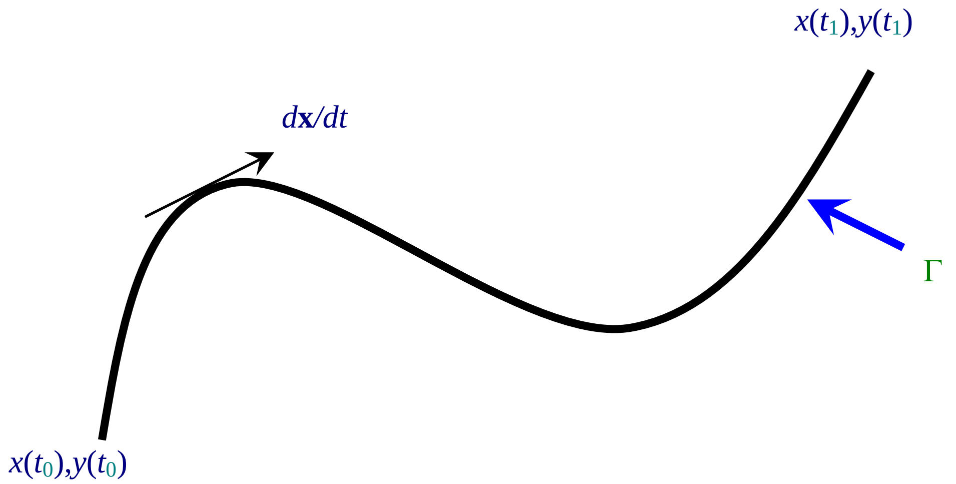
$$I = \int f(x,a)\,dx\,.$$

Other times, however, we might want to evaluate the integral along some other trajectory through the variable space.

Consider the scalar field $f(x,y,z)$ in three-dimensions, and the curve Γ defined parametrically by

$$x = x(t),\ y = y(t),\ z = z(t) \text{ for } t_0 < t < t_1.$$

Each point on the curve $\mathbf{x} = \mathbf{x}(t)$ is associated with a specific value of the parameter t (more than one t could yield the same point $\mathbf{x}$), and each point has its own unique value of $f(\mathbf{x})$.



Note that the vector $\dfrac{d\mathbf{x}}{dt} = \dfrac{dx}{dt}\mathbf{i} + \dfrac{dy}{dt}\mathbf{j} + \dfrac{dz}{dt}\mathbf{k}$ is in the direction tangential to the curve, though its magnitude depends on the precise form of $\mathbf{x}(t)$.

If t represents time, then the curve Γ would be the trajectory of the point $\mathbf{x}(t)$, and $d\mathbf{x}/dt$ would be the velocity. However, often we need to define the curve parametrically where the parameter (t) has **no** particular physical meaning.

One useful (special) choice of the parameter t that does have a physical meaning is the arc length: the distance along the curve. Such a parameter is frequently given the symbol s. In particular,

$$d\mathbf{x} = \frac{d\mathbf{x}}{ds} ds,$$

but if s is the distance along the curve then $|d\mathbf{x}| = ds$, so $|d\mathbf{x}/ds| = 1$, and $d\mathbf{x}/ds$ is a unit vector tangential to the curve.

4.4.1 Line integral of a scalar field

Suppose $\mathbf{x}(t)$ represented the path taken by a lawn mower and the scalar field $\sigma(\mathbf{x})$ represents the length of the grass. An obvious question is: *how full is the lawn mower's grass catcher at the end of the path?*

This will depend on the length of the path and the length of the grass beneath the lawn mower (hopefully there is no grass to be cut when the mower is on hard paving!), but not (within reason) the time taken to complete the path. To reinforce this point, we will us s as the parameter as *distance along the path* when defining the path $\mathbf{x}(s)$, with s varying between s_a and s_b. [We will also assume, for the moment, that the path cut by the lawn mower does not overlap with regions it has already cut.]

The volume of grass cut will naturally depend on the integral of the grass length (times the width of the mower) over the distance along the path, namely

$$I = \int_{s_a}^{s_b} \sigma\left(\mathbf{x}(s)\right) ds .$$

We can express this as a sum as

$$I = \int_{s_a}^{s_b} \sigma\left(\mathbf{x}(s)\right) ds = \lim_{N\to\infty} \sum_{i=0}^{N} \sigma\left(\mathbf{x}_i\right) \left|\mathbf{x}_{i+1} - \mathbf{x}_i\right|$$

where all $x_i = \mathbf{x}(s_i)$ lie on the path with s_i in order from s_a to s_b.

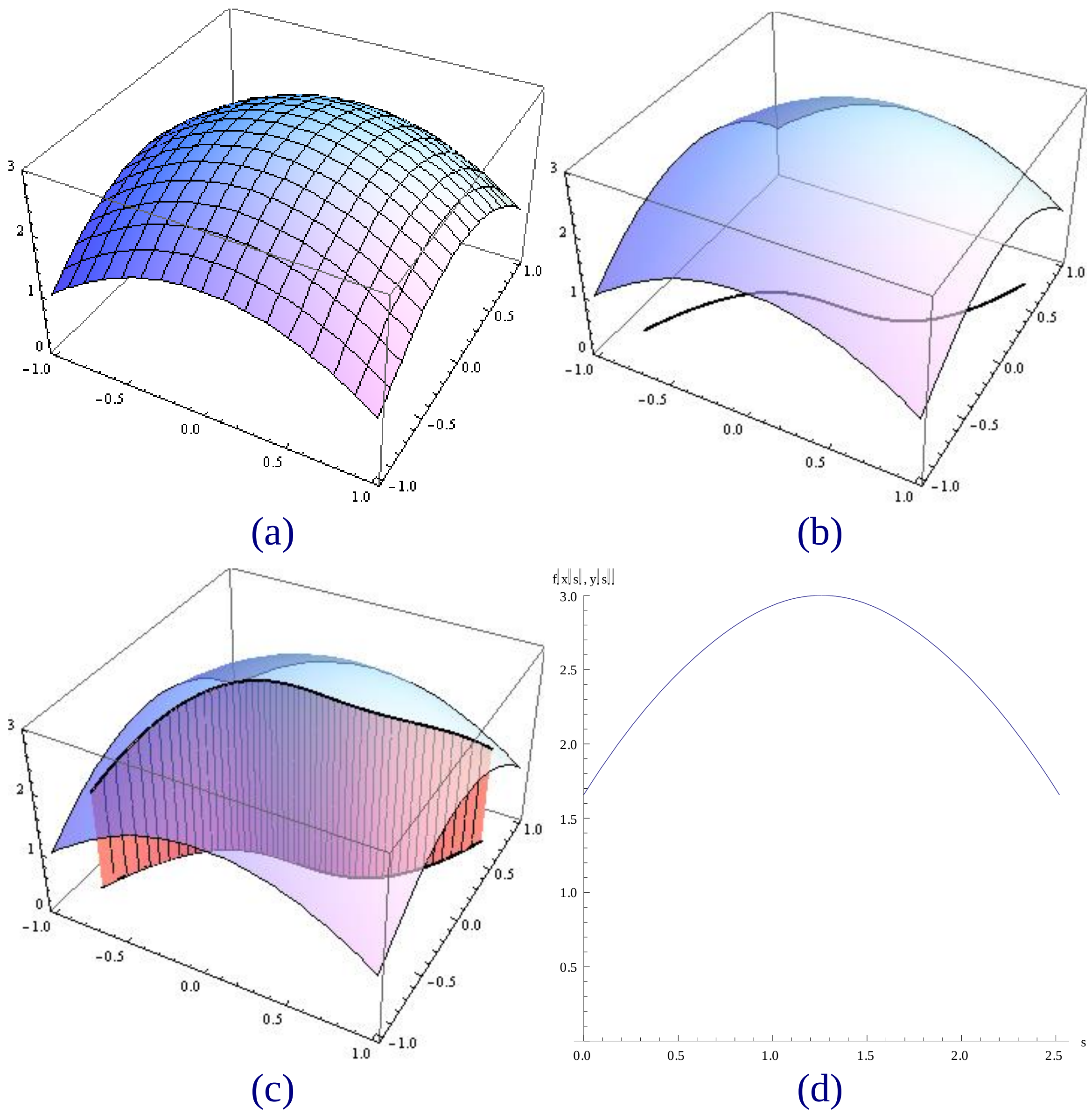


Figure 15: Line integral of scalar field. (a) Scalar field. (b) Path along which integral to be evaluated. (c) Surface representing the integral. (d) Value of scalar field along path expressed as function of distance along the path.

Example 2

Consider the path Γ such that $\mathbf{x}(t) = t\mathbf{i} + (t-1)^2\mathbf{j}$ for $0 \le t \le 2$ through the scalar field $f(x,y) = x + y$. Compute I_t, the integral of $f(x,y)$, along the path.

$$
\begin{aligned}
I_t &= \int_\Gamma f\left(\mathbf{x}(t)\right) dt = \int_0^2 f\left(t,(t-1)^2\right) dt \\
&= \int_0^2 t + (t-1)^2 \, dt = \int_0^2 t^2 - t + 1 \, dt \\
&= \left[\tfrac{1}{3}t^3 - \tfrac{1}{2}t^2 + t\right]_0^2 = \frac{8}{3} - 2 + 2 \\
&= \frac{8}{3}
\end{aligned}
$$

Changing parameterisation

If we change the parameterisation of the path, for example to $\mathbf{x}(s) = 2s\mathbf{i} + (2s-1)^2\mathbf{j}$ with $0 \le s \le 1$, then the result of the integral, I_s, is different:

$$
\begin{aligned}
I_s &= \int_\Gamma f\left(\mathbf{x}(s)\right) ds = \int_0^1 f\left(2s,(2s-1)^2\right) ds \\
&= \int_0^1 2s + (2s-1)^2 \, dt = \int_0^1 4s^2 - 2s + 1 \, ds \\
&= \left[\tfrac{4}{3}s^3 - s^2 + s\right]_0^1 = \tfrac{4}{3} - 1 + 1 \\
&= \frac{4}{3}
\end{aligned}
$$

Line integrals of the form $I = \int_{s_a}^{s_b} \sigma\left(\mathbf{x}(s)\right)\, ds$ depend on the relationship $\mathbf{x}(s)$ as well as the function $\sigma(\mathbf{x})$ being integrated.

In the above example, the parameters t and s re related by $t = 2s$.Thus we can use a change of variables to write

$$I_t = \int_\Gamma f\left(\mathbf{x}(t)\right) dt = \int_\Gamma f\left(\mathbf{x}(s)\right) \frac{dt}{ds}\, ds = 2\int_\Gamma f\left(\mathbf{x}(s)\right) ds$$

4.4.2 Line integral of a vector field

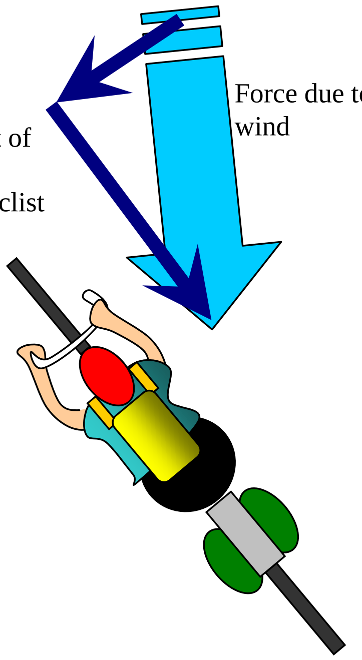
A common type of line integral computes the component of a vector field in the direction of the path.

The wind drag on a cyclist is proportional to the square of the apparent wind speed*, and oriented in the direction of the apparent wind. If the cyclist is moving slowly compared with the wind (and so the cyclists velocity does not affect the apparent window velocity), then we can approximate this force on the cyclist by the vector field

$$\mathbf{F}(\mathbf{x}) = \alpha \left|\mathbf{U}(\mathbf{x})\right| \mathbf{U}(\mathbf{x})$$

where the vector field $\mathbf{U}(\mathbf{x})$ describes the wind field (strength and orientation), and α is a constant.

* You do not need to understand the physics of this problem.



As the cyclist moves forwards, they must overcome (or are helped by) the component of this force aligned with their direction of travel.

Suppose the cyclist has to cover a route Γ described by the curve $\mathbf{x}(s)$ from s_a to s_b. How does the presence of the wind change the amount of energy the cyclist must expend to reach their destination?

The direction the cyclist is moving in when at $\mathbf{x}(s)$ is $d\mathbf{x}/ds$, so the component of the force acting to *help* the cyclist is

$$f\left(\mathbf{x}(s)\right) = \mathbf{F}\left(\mathbf{x}(s)\right) \cdot \frac{d\mathbf{x}}{ds}$$

and the energy gained (positive) or expended (negative) is

$$E = \int_{s_a}^{s_b} f\left(\mathbf{x}(s)\right) ds = \int_{s_a}^{s_b} \mathbf{F}\left(\mathbf{x}(s)\right) \cdot \frac{d\mathbf{x}}{ds}\, ds$$

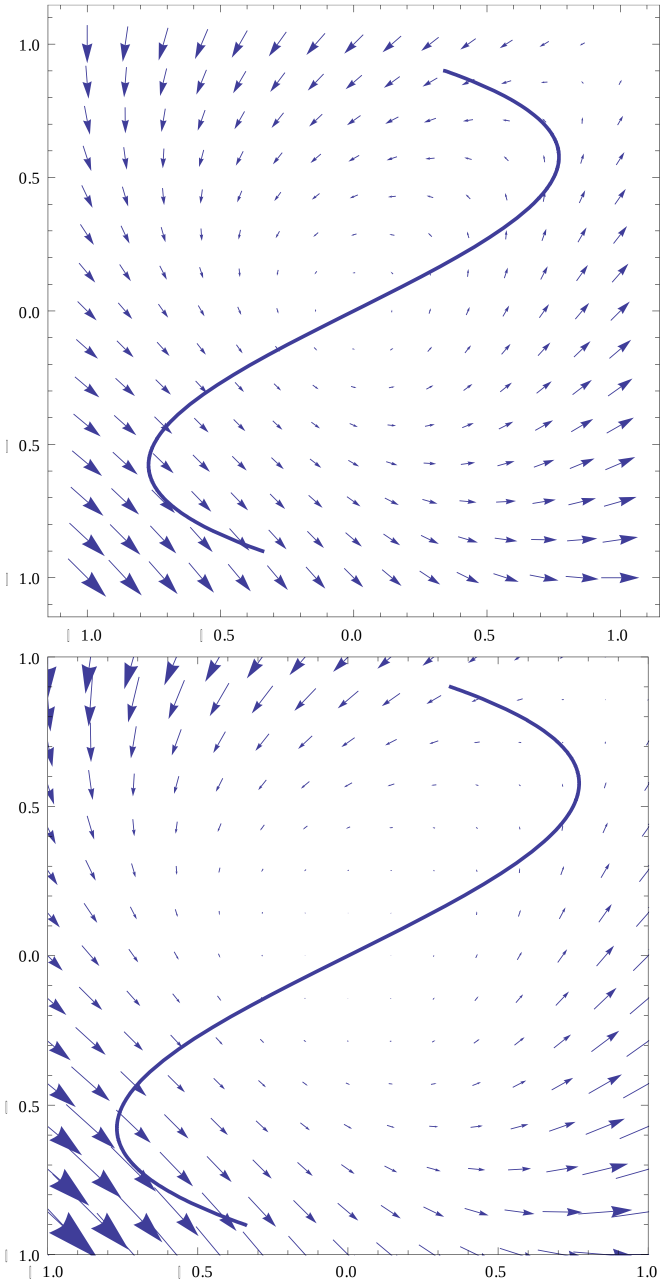


Figure 16: Cycling through wind. (a) The path through the wind field; (b) the path through the force field due to the wind.

Here we have computed the energy as the product of force and distance.

Alternatively, we could have computed the energy (work) as the product of power (rate of work) and time. Doing this is straight forward. Instead of parameterising the path using $\mathbf{x}(s)$, we could parameterise using time as $\mathbf{x}(t)$ with $t_a \le t \le t_b$. Since this choice of parameterisation is arbitrary, so long as it describes the same curve, we know that

$$\frac{d\mathbf{x}}{ds}ds = \frac{d\mathbf{x}}{dt}dt = d\mathbf{x},$$

and so we could write the integral as

$$E = \int_{s_a}^{s_b} \mathbf{F}(\mathbf{x}(s)) \cdot \frac{d\mathbf{x}}{ds} ds = \int_{t_a}^{t_b} \mathbf{F}(\mathbf{x}(t)) \cdot \frac{d\mathbf{x}}{dt} dt = \int_\Gamma \mathbf{F} \cdot d\mathbf{x}.$$

The last of these forms is frequently used because of its compactness and to highlight the fact the integral is independent of the parameterisation. Note that we do not write this as an integral from (x_a,y_a) to (x_b,y_b) as the integral is a function of the path taken rather than just the end points.

Example A

Evaluate $K = \int_\Gamma \mathbf{u} \cdot d\mathbf{x}$ for the vector field $\mathbf{u} = \mathbf{i} + 2y\mathbf{j}$ along the line between the origin and the point $(x,y) = (1,1)$.

We begin by parameterising the path. Suppose we choose the distance along the path, s, as our parameter, then $x(s) = s/\sqrt{2}$ and $y(s) = s/\sqrt{2}$, with $0 \le s \le \sqrt{2}$.

Noting that $dx/ds = 1/\sqrt{2}$ and $dy/ds = 1/\sqrt{2}$, then

$$\begin{aligned}
K &= \int_\Gamma \mathbf{u} \cdot d\mathbf{x} = \int_\Gamma \mathbf{u} \cdot \frac{d\mathbf{x}}{ds} ds \\
&= \int_\Gamma (\mathbf{i} + 2y\mathbf{j}) \cdot \left(\tfrac{1}{\sqrt{2}}\mathbf{i} + \tfrac{1}{\sqrt{2}}\mathbf{j}\right) ds \\
&= \int_\Gamma \left(\tfrac{1}{\sqrt{2}} + \tfrac{2}{\sqrt{2}} y(s)\right) ds \\
&= \int_0^{\sqrt{2}} \left(\tfrac{1}{\sqrt{2}} + s\right) ds \\
&= \left[\tfrac{1}{\sqrt{2}} s + \tfrac{1}{2} s^2\right]_0^{\sqrt{2}} \\
&= 1 + 1 = 2
\end{aligned}$$

Alternative parameterisation

If instead we were to parameterise the path as $x(t) = t^2$ and $y(t) = t^2$ for $0 \le t \le 1$, then $dx/dt = 2t$ and $dy/dt = 2t$ so

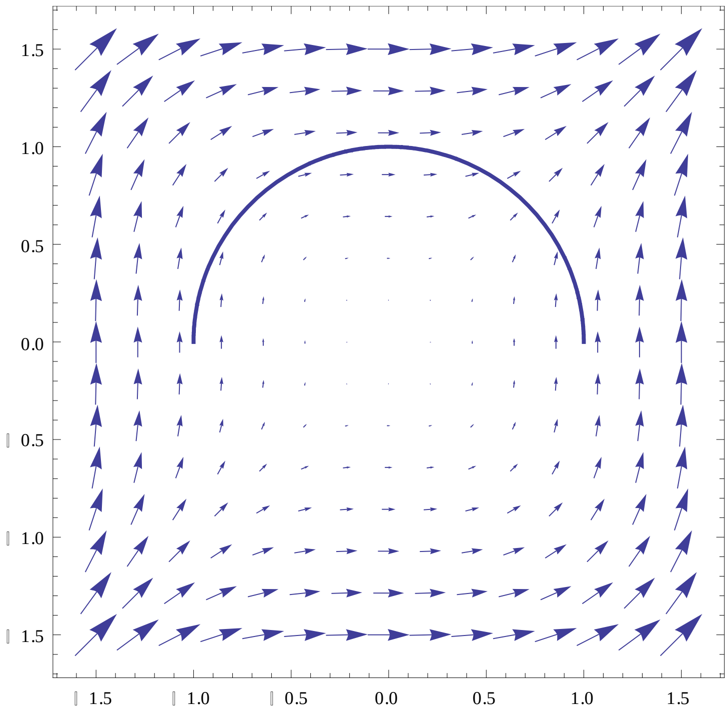
$$
\begin{aligned}
L &= \int_\Gamma \mathbf{u}\cdot d\mathbf{x} = \int_\Gamma \mathbf{u}\cdot\frac{d\mathbf{x}}{dt}\,dt \\
&= \int_\Gamma (\mathbf{i} + 2y\mathbf{j})\cdot(2t\mathbf{i} + 2t\mathbf{j})\,dt \\
&= \int_\Gamma (2t + 4ty(t))\,dt \\
&= \int_0^1 (2t + 4t^3)\,dt \\
&= \left[t^2 + t^4\right]_0^1 \\
&= 1 + 1 = 2
\end{aligned}
$$

Unlike the line integrals of the scalar field, with a vector field we get the same answer regardless of the parameterisation used for the path.

Here, we have worked in Cartesian vectors. It is very much simpler to do this than to work in other coordinate systems, even if the path has a polar parameterisation.

Example B

Evaluate $I_1 = \int_{\Gamma_1} \mathbf{F} \cdot d\mathbf{x}$ for the vector field $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j}$ where the path Γ_1 is the semicircle $x^2 + y^2 = 1$ from (–1,0) to (1,0) for positive y.



We begin by parameterising the semicircle Γ_1 as

$$\mathbf{x}(\theta) = -\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

for $0 \le \theta \le \pi$ (the negative sign in the first component since we traverse the circle clockwise for increasing θ).

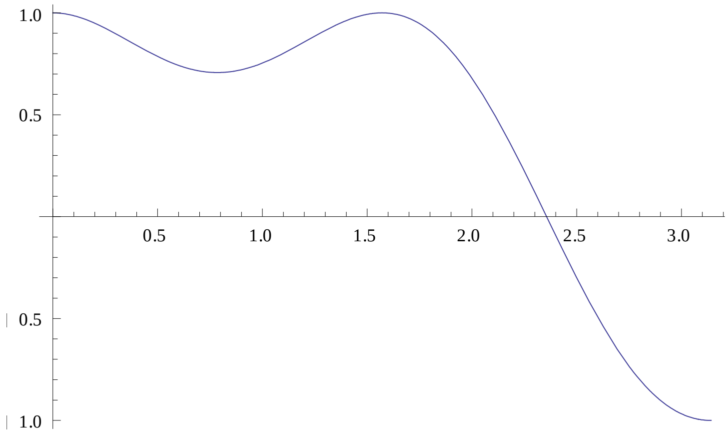
Then
$$\frac{d\mathbf{x}}{d\theta} = \sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$$

and
$$\mathbf{F} = r^2\sin^2\theta\,\mathbf{i} + r^2\cos^2\theta\,\mathbf{j}$$
$$= \sin^2\theta\,\mathbf{i} + \cos^2\theta\,\mathbf{j}$$

on the semicircle as $r = 1$.

What we need to integrate is

$$\mathbf{F} \cdot \frac{d\mathbf{x}}{d\theta} = \sin^3\theta + \cos^3\theta$$



Hence

$$I_1 = \int_{\Gamma_1} \mathbf{F}\cdot d\mathbf{x} = \int_{\theta=0}^{\theta=\pi} \mathbf{F}\cdot\frac{d\mathbf{x}}{d\theta}\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \sin^3\theta + \cos^3\theta\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \sin\theta\left(1-\cos^2\theta\right) + \cos\theta\left(1-\sin^2\theta\right)d\theta$$

$$= \left[-\cos\theta + \frac{1}{3}\cos^3\theta + \sin\theta - \frac{1}{3}\sin^3\theta\right]_0^{\pi}$$

$$= 2\left[1-\frac{1}{3}\right] = \frac{4}{3}$$

Note that if we reversed the direction we moved along the curve then we would reverse the sign of the integral: effectively we swap the upper and lower integration limits on the parameter.

Alternative path

Now consider the integral $I_2 = \int_{\Gamma_2} \mathbf{F}\cdot d\mathbf{x}$ with the same vector field $\mathbf{F}(\mathbf{x})$ but following the path Γ_2 that is a straight line from (–1,0) to (1,0).

Parameterise the path Γ_2:

$$\mathbf{x}(t) = (t-1)\,\mathbf{i} \quad \text{for} \quad 0 \le t \le 2.$$

Hence
$$\frac{d\mathbf{x}}{dt} = \mathbf{i}$$

and
$$\mathbf{F} = x^2\,\mathbf{j} = \left(t-1\right)^2\,\mathbf{j}$$

on the line.

What we need to integrate is

$$\mathbf{F}\cdot\frac{d\mathbf{x}}{dt}=0$$

so clearly $$I_2=\int_{\Gamma_2}\mathbf{F}\cdot d\mathbf{x}=\int_{\Gamma_2}0\,dt=0$$

and $I_1 \neq I_2$.

The difference between I_1 and I_2 in the above example illustrates clearly that the value of a line integral depends on the entire curve and not just on the end points.

Moreover, if we define a closed curve Γ_0 defined as Γ_1 then Γ_2 in reverse ($-\Gamma_2$), then the integral around this entire curve (regardless of the starting point is

$$I_{1\text{-}2} = I_1 - I_2 = 4/3.$$

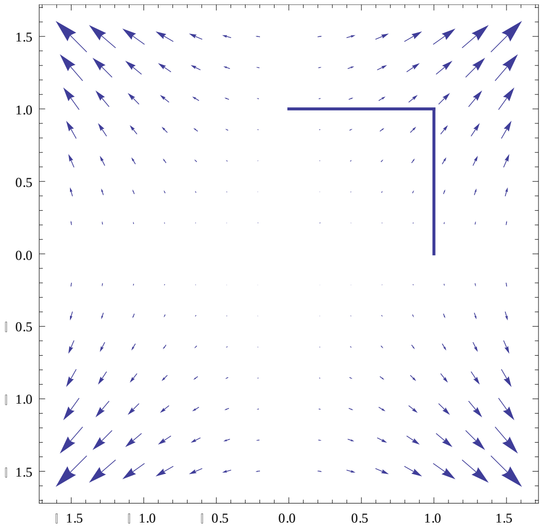
We write such an integral around a closed loop as

$$\oint_{\Gamma_0}\mathbf{F}\cdot d\mathbf{x}=\int_{\Gamma_1}\mathbf{F}\cdot d\mathbf{x}-\int_{\Gamma_2}\mathbf{F}\cdot d\mathbf{x}=\frac{4}{3}.$$

As before, reversing the direction of propagation around the circuit will reverse the sign of the result.

Example B

Evaluate $J_1 = \int_{\Gamma_1} \mathbf{G}\cdot d\mathbf{x}$, where $\mathbf{G} = \alpha x y^2\mathbf{i} + \beta x^2 y\mathbf{j}$ and Γ_1 is the path made up of straight lines from (1,0) to (1,1) and then from (1,1) to (0,1).



We begin by dividing the path Γ_1 into two parts, Γ_{1a} and Γ_{1b}, corresponding to the two line segments.

Parameterise Γ_{1a} as $\mathbf{x}_a(s) = \mathbf{i} + s\mathbf{j}$ for $0 \le s \le 1$,

and Γ_{1b} as $\mathbf{x}_b(t) = (1 - t)\mathbf{i} + \mathbf{j}$ for $0 \le t \le 1$.

Now
$$\frac{d\mathbf{x}_a}{ds} = \mathbf{j}, \quad \frac{d\mathbf{x}_b}{dt} = -\mathbf{i}$$

along which
$$\mathbf{G}(\mathbf{x}_a(s)) = \alpha s^2\mathbf{i} + \beta s\mathbf{j}, \quad \mathbf{G}(\mathbf{x}_b(t)) = \alpha(1-t)\mathbf{i} + \beta(1-t)^2\mathbf{j}$$

and so
$$\mathbf{G}(\mathbf{x}_a)\cdot\frac{d\mathbf{x}_a}{ds} = \beta s, \quad \mathbf{G}(\mathbf{x}_b)\cdot\frac{d\mathbf{x}_b}{dt} = -\alpha(1-t)$$

giving

$$
\begin{aligned}
J_1 = \int_{\Gamma_1} \mathbf{G} \cdot d\mathbf{x} &= \int_{\Gamma_{1a}} \mathbf{G} \cdot d\mathbf{x} + \int_{\Gamma_{1b}} \mathbf{G} \cdot d\mathbf{x} \\
&= \int_0^1 \mathbf{G}(\mathbf{x}_a) \cdot \frac{d\mathbf{x}_a}{ds}\, ds + \int_0^1 \mathbf{G}(\mathbf{x}_b) \cdot \frac{d\mathbf{x}_b}{dt}\, dt \\
&= \int_0^1 \beta s - \alpha(1-s)\, ds \\
&= \left[\tfrac{1}{2}\beta s^2 - \tfrac{1}{2}\alpha(1-s)^2 \right]_0^1 \\
&= \tfrac{1}{2}(\beta - \alpha)
\end{aligned}
$$

Alternate path

Now consider $J_2 = \int_{\Gamma_2} \mathbf{G} \cdot d\mathbf{x}$ with the same $\mathbf{G}$ as before, but with Γ_2 representing a straight line from (1,0) to (0,1):

$$\mathbf{x}_2(t) = (1-t)\mathbf{i} + t\mathbf{j} \quad \text{for} \quad 0 \le t \le 1.$$

$$\Rightarrow \qquad \frac{d\mathbf{x}_2}{dt} = -\mathbf{i} + \mathbf{j}$$

$$
\begin{aligned}
\mathbf{G}(\mathbf{x}_2(t)) &= \alpha x y^2 \mathbf{i} + \beta x^2 y \mathbf{j} \\
&= \alpha(1-t)t^2 \mathbf{i} + \beta(1-t)^2 t \mathbf{j}
\end{aligned}
$$

$$\mathbf{G}(\mathbf{x}_2) \cdot \frac{d\mathbf{x}_2}{dt} = -\alpha(1-t)t^2 + \beta(1-t)^2 t$$

$$J_2 = \int_{\Gamma_2} \mathbf{G} \cdot d\mathbf{x} = \int_0^1 -\alpha(1-t)t^2 + \beta(1-t)^2\, t\, dt$$
$$= \int_0^1 -\alpha\left(t^2 - t^3\right) + \beta\left(t - 2t^2 + t^3\right) dt$$
$$= \left[-\alpha\left(\tfrac{1}{3}t^3 - \tfrac{1}{4}t^4\right) + \beta\left(\tfrac{1}{2}t^2 - \tfrac{2}{3}t^3 + \tfrac{1}{4}t^4\right)\right]_0^1$$
$$= \frac{1}{12}(\beta - \alpha)$$

Hence, $J_2 \neq J_1$ except in the special case when $\beta = \alpha$, *i.e.*,

$$J_1 \equiv \frac{1}{2}(\beta-\alpha) \;\neq\; J_2 \equiv \frac{1}{12}(\beta-\alpha)$$

In the second example, as with the first, although the start and end points of the curves Γ_1 and Γ_2 are the same, the integrals are not equal.

4.5 Conservative vector fields

In the examples we have considered so far, the line integral has depended on the path taken as well as the end points. However, this is not always the case, as was illustrated by the second example when $\alpha = \beta$.

Consider a vector field $\mathbf{F}(\mathbf{x})$ that can be expressed as the gradient of a scalar field. In particular, consider

$$\mathbf{F}(\mathbf{x}) = \nabla\varphi(\mathbf{x}).$$

Not all vector fields can be written in this way. As we shall see, those that can be have special properties. The scalar field $\varphi(\mathbf{x})$ is often referred to as a *scalar potential*, or sometimes just a *potential* field.

Forces can often be written in this way*. Physical examples include the pressure field p, the gradient of which acts as a force on fluid particles. This force acts in the opposite direction to the gradient, hence the inclusion of the negative sign in the above relationship.

Suppose we want to consider the work done against the force by moving along a path Γ parameterised as $\mathbf{x}(t)$ for $t_0 \le t \le t_1$. As we have seen before, we can write this as

$$\begin{aligned}
\int_\Gamma \mathbf{F}\cdot d\mathbf{x} &= \int_\Gamma \nabla\varphi\cdot d\mathbf{x} \\
&= \int_{t_0}^{t_1} \nabla\varphi\cdot\frac{d\mathbf{x}}{dt}\,dt \\
&= \int_{t_0}^{t_1}\left(\frac{\partial\varphi}{\partial x}\frac{dx}{dt}+\frac{\partial\varphi}{\partial y}\frac{dy}{dt}+\frac{\partial\varphi}{\partial z}\frac{dz}{dt}\right)dt \\
&= \int_{t_0}^{t_1}\frac{d}{dt}\varphi\left(\mathbf{x}(t)\right)dt \\
&= \left[\varphi\left(\mathbf{x}(t)\right)\right]_{t_0}^{t_1} = \varphi\left(\mathbf{x}(t_1)\right)-\varphi\left(\mathbf{x}(t_0)\right) \\
&= \varphi\left(\mathbf{x}_1\right)-\varphi\left(\mathbf{x}_0\right)
\end{aligned}$$

In contrast with the previous examples, we have found a result without having to specify the path other than its start and end points.

By considering any two points $\mathbf{x}_a$ and $\mathbf{x}_b$ on a closed loop Γ, and noting that the line integral from $\mathbf{x}_a$ to $\mathbf{x}_b$ is negative that from $\mathbf{x}_b$ to $\mathbf{x}_a$, then it is obvious that the integral around a closed loop vanishes, *i.e.*

$$\oint_\Gamma \mathbf{F}\cdot d\mathbf{x} = \oint_\Gamma \nabla\varphi\cdot d\mathbf{x} = 0.$$

---

* For physical sciences, we will often define the scalar potential so that $\mathbf{F}(\mathbf{x}) = -\nabla\varphi(\mathbf{x})$. Thus, for example, the force due to pressure is directed from high pressure to low pressure. For the purposes of this *mathematical* discussion, we will take $\mathbf{F}(\mathbf{x}) = \nabla\varphi(\mathbf{x})$ to avoid having to carry around extra minus signs.

Any field $\mathbf{F}(\mathbf{x})$ that has this property is called a *conservative field*, reflecting the fact the result of a line integral is preserved independently of the path taken.

Note that in some cases the field may be conservative over only part of the domain. We shall look at an example of this shortly.

Exact differentials

The idea of conservative fields is related to that of exact differentials. In particular, if

$$\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$$

then

$$\mathbf{F}\cdot d\mathbf{x} = P(x,y)dx + Q(x,y)dy.$$

If this is an *exact differential* then we can write

$$\mathbf{F}\cdot d\mathbf{x} = P(x,y)dx + Q(x,y)dy = d\varphi,$$

so that

$$\int_\Gamma \mathbf{F}\cdot d\mathbf{x} = \int_\Gamma d\varphi = [\varphi]_{\mathbf{x}_0}^{\mathbf{x}_1} = \varphi(\mathbf{x}_1) - \varphi(\mathbf{x}_0)$$

and the integral depends on the end points $\mathbf{x}_0$ and $\mathbf{x}_1$ but not the path Γ taken. For $\mathbf{F}\cdot d\mathbf{x}$ to be exact we require

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0.$$

Consider a *vector field* expressed as the *gradient of a scalar field*, $\mathbf{F} = \nabla\varphi$, then

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial y}\right) = 0$$

and the differential is exact so we can write $\mathbf{F}\cdot d\mathbf{x} = d\varphi$ and the integral is independent of the path.

Example A

Earlier we considered $\mathbf{G} = \alpha xy^2\mathbf{i} + \beta x^2y\mathbf{j}$, so

$$\frac{\partial P}{\partial y} = 2\alpha xy \quad \text{and} \quad \frac{\partial Q}{\partial x} = 2\beta xy,$$

thus **G** is conservative if and only if $\alpha = \beta$, which is the same condition as we saw for the integral in the example to be independent of path.

A test to see if field is conservative

We have defined a conservative field as $\mathbf{F} = \nabla\varphi$, but if we are simply given **F** and not φ, how can we tell whether or not **F** is conservative? As we have just seen, for a two-dimensional field, we could use the condition that **F** must be an exact differential. However, we can generalise this approach using one of the two identities we discussed in §4.3.4. In particular, we saw that

$$\nabla\times\nabla\varphi = \mathbf{0}$$

(a zero vector) for any (continuous, differentiable) scalar field φ. Thus, if $\mathbf{F} = \nabla\varphi$ then we must have

$$\nabla\times\mathbf{F} = \mathbf{0}$$

if **F** is conservative (and that **F** is irrotational).

This is one way to test if a given vector field is conservative. For a two-dimensional field, it is equivalent to the test for an exact differential introduced above. For a three-dimensional field, we require all three components of $\nabla\times\mathbf{F}$ to vanish. *This is the sort of thing examiners like to ask!*

Conservative force

A *conservative force* is one that can be written as the gradient of a potential. By convention, we will generally define the potential so that

$$\mathbf{F} = -\nabla\varphi.$$

This means, amongst other things, that if **F** is the force then φ is the *potential energy*. [If a particle gains potential energy moving from $\mathbf{x}_0$ to $\mathbf{x}_1$ then the word done by the force on the particle must be negative.]

In particular, a conservative force does no work if a particle moves around a closed path and returns to its initial position.

Since the curl of a gradient is automatically zero (one of the vector identities we looked at in §4.3.4), then $-\nabla\times\nabla\varphi = \nabla\times\mathbf{F} = \mathbf{0}$, as expected.

Our choice earlier to omit the minus sign and use $\mathbf{F} = \nabla\varphi$ does not change any of these arguments.

More properties of conservative fields

We have seen that $\mathbf{F} = \nabla\varphi$ implies that $\oint_\Gamma \mathbf{F}\cdot d\mathbf{x} = 0$ for any closed curve Γ.

Conversely, if $\oint_\Gamma \mathbf{F}\cdot d\mathbf{x} = 0$ for **all** closed curves Γ, then this implies $\mathbf{F} = \nabla\varphi$.

The condition required for this converse is very strong: it is not sufficient that $\oint_\Gamma \mathbf{F}\cdot d\mathbf{x} = 0$ for a single or even multiple closed curves.

Multi-valued functions
important to know but not to replicate

Care must be taken if $\varphi(\mathbf{x})$ is not single-valued. For example, consider $\varphi = \tan^{-1}(y/x)$ since $\tan(\varphi + n\pi) = y/x$ for any integer n. In particular

$$\frac{\partial\varphi}{\partial x} = -\frac{y}{x^2}\frac{1}{1+y^2/x^2} = -\frac{y}{x^2+y^2} \quad \text{and} \quad \frac{\partial\varphi}{\partial y} = \frac{x}{x^2+y^2}.$$

Since $\mathbf{F} = \nabla\varphi$, then we expect the line integral around all closed curves to be zero. However, consider the integral around a circle of radius a centred on the origin such that $\mathbf{x}(\theta) = a(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$, so

$$\frac{d\mathbf{x}}{d\theta} = a\left(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\right)$$

and

$$\mathbf{F} = \left(-\frac{1}{r}\sin\theta\,\mathbf{i} + \frac{1}{r}\cos\theta\,\mathbf{j}\right)_{r=a} = \frac{1}{a}\left(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\right)$$

giving

$$\begin{aligned}\oint_\Gamma \mathbf{F}\cdot d\mathbf{x} &= \int_0^{2\pi}\left[\frac{1}{a}\left(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\right)\right]\cdot\left[a\left(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\right)\right]d\theta \\ &= \int_0^{2\pi}\left(\sin^2\theta + \cos^2\theta\right)d\theta \\ &= \int_0^{2\pi} d\theta \\ &= 2\pi\end{aligned}$$

which is independent of a but not zero, even though the start and end points are the same and the field is conservative… except that there is a *singularity* at the origin.

4.6 Surface integrals

In §3.2 we considered double integrals as the integral of a function with two independent variables over an area, while in §4.4 we have considered integrals along a line of a function of two (or more) independent variables. An obvious extension to this idea is to consider the integral over an area for a function of three independent variables.

Two-dimensional surface

A finite region S on a plane surface is two-dimensional. Its key characteristics are (a) its area A, and (b) its orientation given by the normal **n** to the surface.

Note that there are two possible (*antiparallel*) orientations for the normal.

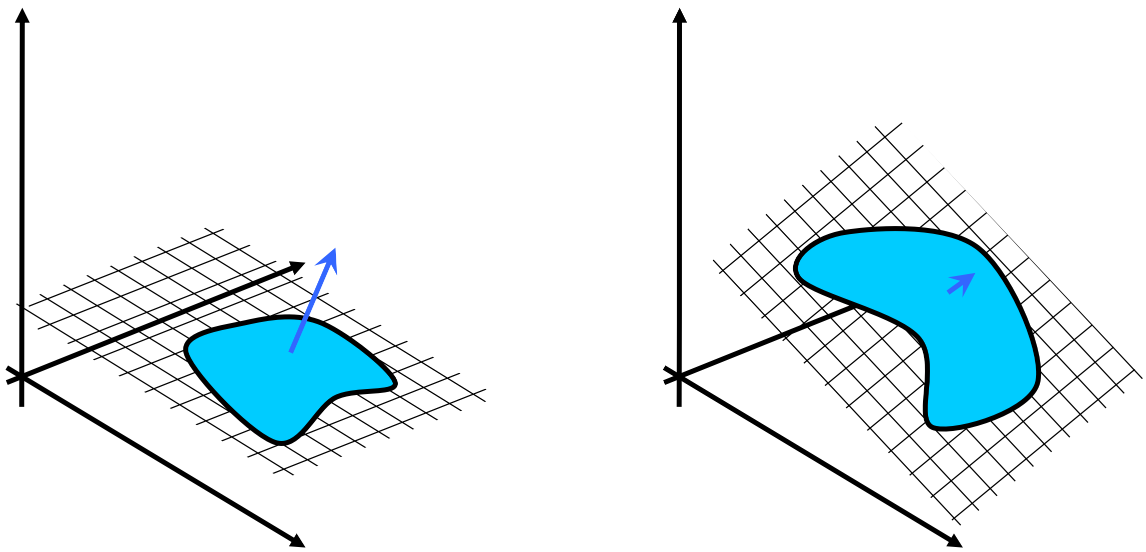


Figure 17: Sketch of regions on plane surfaces.

Three-dimensional surface

We can generalise these ideas to a region on a three-dimensional curved surface.

Consider a volume V in three-dimensions, bounded by a (curved) surface S and take a small region A of the surface S.

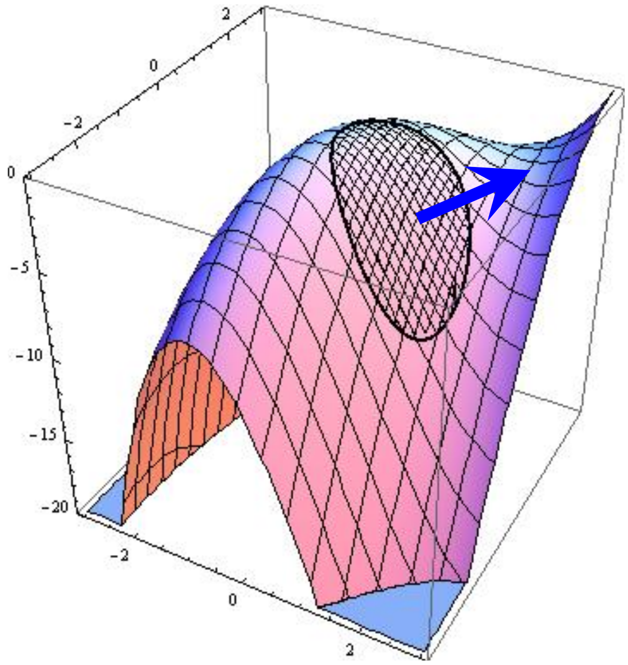


Figure 18: Region of a surface showing normal vector.

Since the region A is small, it appears to be a plane and can be considered to have (a) and area *dS* (say) and (b) an orientation defined by the vector **n**. We take **n** to be the unit normal to S in the small region A. By convention we define **n** as the *outward normal* from V.

4.6.1 Vector area

We can define the *vector area*

$$\mathbf{dS} = \mathbf{n}dS$$

to represent the small region (A) of S in the limit of the area of this region tending to zero.

If we integrate **dS** over a finite *plane region* S then

$$\int_S \mathbf{dS} = \int_S \mathbf{n}\, dS = \mathbf{n}\int_S dS = A\mathbf{n}$$

where *A* is the area of the region.

If we integrate the vector area over a closed surface then there is cancellation from the contributions from the 'opposite sides'. Thus for any close surface S,

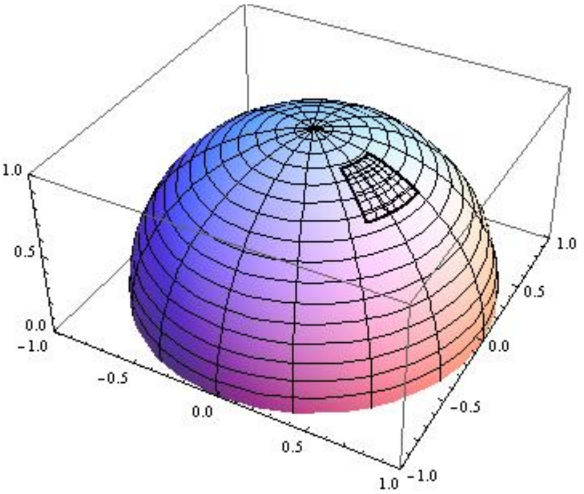
$$\int_S \mathbf{dS} = \mathbf{0},$$

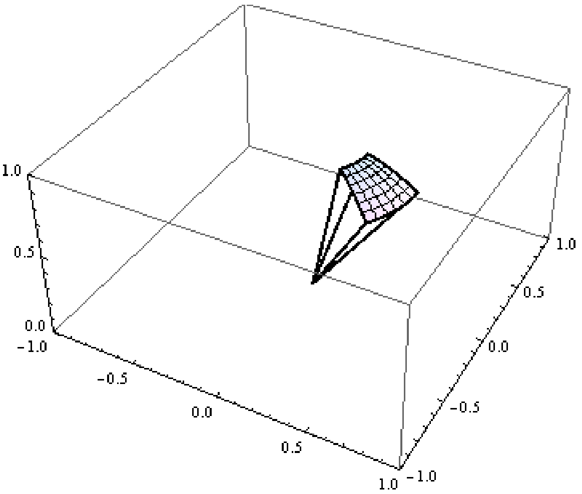
a zero vector.

Vector area of hemisphere

Calculate the vector area of the hemispherical surface S given by $x^2 + y^2 + z^2 = a^2$ and $z > 0$.

It is useful to use spherical polar coordinates (r,θ,ϕ) to describe variations on the surface, but best to write the vectors in terms of the Cartesian unit vectors **i**, **j** and **k** in the x, y and z directions, respectively.





The hemisphere has $r = a$ with $0 \le \theta \le \pi/2$ and $-\pi \le \phi \le \pi$.

Using spherical polar coordinates as parameters, we can define the surface parametrically as

$\mathbf{x} = \mathbf{x}(\theta,\phi) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = a \sin\theta\cos\phi\,\mathbf{i} + a \sin\theta\sin\phi\,\mathbf{j} + a \cos\theta\,\mathbf{k}$.

For small variations $d\theta$ and $d\phi$ in θ and ϕ, the scalar element area centred at $\mathbf{x}(\theta,\phi)$ as size $dS = (a\sin\theta\, d\phi)(a\, d\theta) = a^2 \sin\theta\, d\theta\, d\phi$.

The normal vector is

$$\mathbf{n} = \frac{\mathbf{x}}{|\mathbf{x}|} = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}$$

$$\int_S d\mathbf{S} = \int_S \mathbf{n}\, dS = \int_{\phi=-\pi}^{\pi}\int_{\theta=0}^{\theta=\pi/2} (\sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k})\, a^2 \sin\theta\, d\theta\, d$$

$$= \int_{\theta=0}^{\theta=\pi/2} a^2 \sin\theta \left[\int_{\phi=-\pi}^{\pi} (\sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k})\ d\phi\right] d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2} a^2 \sin\theta \left[\sin\theta\sin\phi\mathbf{i} - \sin\theta\cos\phi\mathbf{j} + \phi\cos\theta\mathbf{k}\right]_{-\pi}^{\pi}\ d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2} 2\pi a^2 \sin\theta\cos\theta\mathbf{k}\ d\theta = 2\pi a^2 \left[-\tfrac{1}{2}\cos^2\theta\right]_0^{\pi/2}$$

$$= \pi a^2 \mathbf{k}$$

Note that this is simply the area when we look in the direction **k**.

If we were to look at the hemisphere in the **i** or **j** directions, then the front and back faces would cancel each other out.

4.6.2 Variations on a surface

If we have a scalar field $\varphi(\mathbf{x})$, there are other forms of integration over a surface we might be interested in.

Mean temperature

If $T(\mathbf{x}) = x^2 + y^2$ is the temperature field, then what is the mean temperature over a spherical shell of radius a ?

The mean temperature is given by

$$\bar{T} = \int_S T(\mathbf{x})\, dS \Big/ \int_S dS .$$

Again, this will be easier in spherical polar coordinates, so we take

$$dS = (a\sin\theta\, d\phi)(a\, d\theta) = a^2 \sin\theta\, d\theta\, d\phi$$

and write

$$T(\mathbf{x}) = a^2 \sin^2\theta\cos^2\phi\ + a^2 \sin^2\theta\sin^2\phi\ = a^2 \sin^2\theta$$

so that

$$\int_S T(\mathbf{x})\,dS = \int_{\phi=-\pi}^{\pi}\int_{\theta=0}^{\pi}\left(a^2\sin^2\theta\right)a^2\sin\theta\,d\theta\,d\phi$$

$$= 2\pi a^4\int_{\theta=0}^{\pi}\sin^3\theta\,d\theta$$

$$= 2\pi a^4\int_{\theta=0}^{\pi}\sin\theta\left(1-\cos^2\theta\right)d\theta$$

$$= 2\pi a^4\left[-\cos\theta+\frac{1}{3}\cos^3\theta\right]_0^{\pi}$$

$$= \frac{8}{3}\pi a^4$$

and $$\int_S dS = 4\pi a^2.$$

Hence, $$\bar{T} = \int_S T(\mathbf{x})\,dS \Big/ \int_S dS = \frac{2}{3}a^2.$$

Force due to pressure

Suppose the pressure on the outside of hemispherical dome $z \geq 0$ of radius a is given by $p(\mathbf{x}) = p_0 - \rho g z$. Determine the net force on the dome.

The pressure exerts a force on the dome in the direction of its local normal. Thus, the resulting force over the dome is given by

$$\mathbf{F} = \int_S p(\mathbf{x})\,\mathbf{dS} = \int_S p(\mathbf{x})\,\mathbf{n}\,dS$$

on the hemisphere. In spherical polar coordinates,

$$p(\mathbf{x}) = p_0 - \rho g z = p_0 - \rho g a\cos\theta.$$

For brevity, we take $p_0 = 0$ (the contribution from non-zero p_0 would be $\mathbf{F} = p_0\pi a^2\mathbf{k}$) and the integral becomes

$$\begin{aligned}
\mathbf{F} &= \int_{\mathsf{S}} p(\mathbf{x})\mathbf{n}\,dS \\
&= \int_{\phi=-\pi}^{\pi}\int_{\theta=0}^{\pi/2} (-\rho g a\cos\theta)(\sin\theta\cos\phi\mathbf{i}+\sin\theta\sin\phi\mathbf{j}+\cos\theta\mathbf{k})a^2\sin\theta\,d\theta\,d\phi \\
&= -\rho g a^3 \int_{\phi=-\pi}^{\pi}\int_{\theta=0}^{\pi/2} \left(\sin^2\theta\cos\theta\cos\phi\mathbf{i}+\sin^2\theta\cos\theta\sin\phi\mathbf{j}+\sin\theta\cos^2\theta\mathbf{k}\right)d\theta\,d\phi \\
&= -\frac{\rho g a^3}{3}\int_{\phi=-\pi}^{\pi}\left[\sin^3\theta\cos\phi\mathbf{i}+\sin^3\theta\sin\phi\mathbf{j}-\cos^3\theta\mathbf{k}\right]_{\theta=0}^{\pi/2}d\phi \\
&= -\frac{\rho g a^3}{3}\int_{\phi=-\pi}^{\pi}(\cos\phi\mathbf{i}+\sin\phi\mathbf{j}+\mathbf{k})\,d\phi \\
&= -\frac{2}{3}\rho g\pi a^3\mathbf{k}
\end{aligned}$$

4.6.3 Flux across surface

The *flux* of a vector field **F** across a surface element with vector area **dS** = **n***dS* is defined as

$$\mathbf{F}\cdot\mathbf{dS} \text{ or } \mathbf{F}\cdot\mathbf{n}dS,$$

i.e., the component of the vector field normal to the surface, multiplied by the area of the element.

Hence the total flux across the surface S is the sum of the fluxes across all the surface elements that make up S, *i.e.*

$$\int_{\mathsf{S}}\mathbf{F}\cdot\mathbf{dS} \equiv \int_{\mathsf{S}}\mathbf{F}\cdot\mathbf{n}\,dS\,.$$

[Once again, we can, of course, define this as the limit of a sum $\lim_{N\to\infty}\sum_{i=1}^{N}\mathbf{F}(\mathbf{x}_i)\cdot\mathbf{n}_i\delta S$.]

As a physical example of where this type of formulation is of value, consider a gas moving with constant velocity **U**. Suppose that the average density of the gas is ρ_0 and we wish to calculate the mass that moves across a two-dimensional region A of area A in a time δt. The unit normal of A is **n**.

The particles that cross A during the interval δt lie in a three-dimension region V that is swept out by moving A through a displacement $-\mathbf{U}\delta t$. To determine the volume, of V, we need to project the displacement $\mathbf{U}\delta t$ onto the normal **n**.

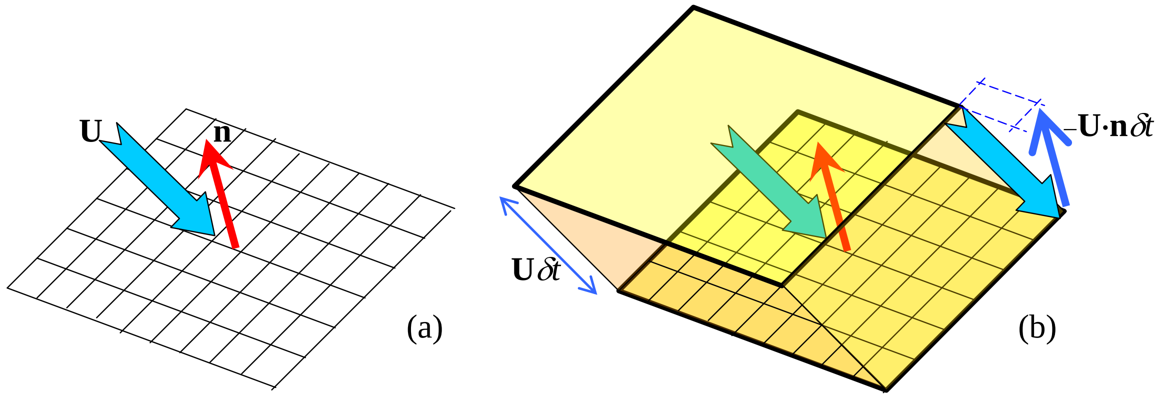


Figure 19: Sketch of flux through a plane surface. (a) Definition of surface, unit normal **n** and velocity field **U**. (b) Illustration of volume that will pass through the surface in a time δt.

The volume is therefore $V = |\mathbf{U}\cdot\mathbf{n}A\delta t|$ and the mass $M = |\rho_0\mathbf{U}\cdot\mathbf{n}A\delta t|$

Often we will be interested in the sign of the flux, not just the mass that passes through the surface in an interval δt. We can define such a *mass flux* across the surface A as

$$Q = V = \rho_0\mathbf{U}\cdot\mathbf{n}A.$$

Non-uniform velocity

We can generalise this idea to cases where the velocity is not constant, but is rather is the vector field $\mathbf{U}(\mathbf{x})$, and the density varies with space as the scalar field $\rho(\mathbf{x})$. We then compute the mass flux across the planar region A as

$$Q = \int_{\mathrm{A}} \rho(\mathbf{x})\mathbf{U}(\mathbf{x})\cdot\mathbf{n}\, dS\,.$$

The region A need not be planar; if it is not, then the direction of the normal changes as a function of **x** across the region, so it would be appropriate to write

$$Q = \int_{A} \rho(\mathbf{x}) \mathbf{U}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, dS\,.$$

There are of course many other physical phenomena that require similar calculations.

Example A

Calculate the integral $J = \int_{S} \mathbf{F} \cdot \mathbf{dS}$, where $\mathbf{F} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$ and S is the hemisphere $x^2 + y^2 + z^2 = a^2$, $z > 0$.

From the earlier example, the hemisphere is given by

$\mathbf{x} = \mathbf{x}(\theta,\phi) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = a\sin\theta\cos\phi\,\mathbf{i} + a\sin\theta\sin\phi\,\mathbf{j} + a\cos\theta\,\mathbf{k}$

with $0 \le \theta \le \pi/2$ and $-\pi \le \phi \le \pi$, and the surface element in spherical polar coordinates is

$$dS = (a\sin\theta\, d\phi)(a\, d\theta) = a^2 \sin\theta\, d\theta\, d\phi.$$

The outward normal for the hemisphere is

$$\mathbf{n} = \frac{\mathbf{x}}{|\mathbf{x}|} = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}$$

so $\mathbf{n}dS = \mathbf{dS} = (\sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k})\, a^2 \sin\theta\, d\theta\, d\phi$

and $\mathbf{F} \cdot \mathbf{dS} = a^2 (\alpha\sin\theta\cos\phi + \beta\sin\theta\sin\phi + \gamma\cos\theta)\sin\theta\, d\theta\, d\phi$

$$\int_S \mathbf{F}\cdot\mathbf{dS} = \int_{\theta=0}^{\theta=\pi/2} a^2 \sin\theta \left[\int_{\phi-\pi}^{\phi=\pi} (\alpha\sin\theta\cos\phi + \beta\sin\theta\sin\phi + \gamma\cos\theta)\, d\phi \right] d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2} a^2 \sin\theta [\alpha\sin\theta\sin\phi - \beta\sin\theta\cos\phi + \gamma\phi\cos\theta]_{\phi=-\pi}^{\phi=\pi}\, d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2} 2\gamma\pi a^2 \sin\theta\cos\theta\, d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2} 2\gamma\pi a^2 \sin\theta\cos\theta\, d\theta$$

$$= \gamma\pi a^2$$

Note that we could have got to this result much faster using our earlier result for $\int_S \mathbf{dS}$ since **F** is a constant vector, so

$$\int_S \mathbf{F}\cdot\mathbf{dS} = \mathbf{F}\cdot\int_S \mathbf{dS} = (\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k})\cdot\left(\pi a^2\mathbf{k}\right) = \gamma\pi a^2.$$

For this particular example, it is clear that the flux across the hemisphere is equal to the flux across the disk $x^2 + y^2 \leq a^2$, $z = 0$. This is a consequence of **F** being constant (so that the flux integral is the dot product of the flux and the vector area), and of the vector area of a closed surface being zero (hence the vector area of the disk is equal in magnitude to the vector area of the hemisphere).

Example B

If the concentration $C(\mathbf{x})$ of a nutrient is given by $C(\mathbf{x}) = -½(\alpha x^2 + \beta y^2 + \gamma z^2)$, calculate the diffusive flux, $\mathbf{F} = -\kappa\nabla C$, across the surface S of an organism, where S is the surface of a circular cylinder of radius a and height $2h$, aligned with the z-axis. The cylinder is centred on the origin.

We begin by noting that (for a diffusivity $\kappa = 1$ in suitable units), the diffusive flux is

$$\mathbf{F} = -\nabla C = (\alpha x\mathbf{i} + \beta y\mathbf{j} + \gamma z\mathbf{k}).$$

In cylindrical polar coordinates, the surface of the cylinder has three parts:

(a) The curved walls S_a: $r = a$ for $0 \le \phi \le 2\pi$, $-h \le z \le h$;

$\Rightarrow$
$$dS = a\, d\phi\, dz$$
$$\mathbf{n} = (x/a)\mathbf{i} + (y/a)\mathbf{j}$$

(b) The bottom S_b: $r \le a$ with $0 \le \phi \le 2\pi$ and $z = -h$

$\Rightarrow$
$$dS = r\, dr\, d\phi$$
$$\mathbf{n} = -\mathbf{k}$$

(c) The top S_c: $r \le a$ with $0 \le \phi \le 2\pi$ and $z = h$

$\Rightarrow$
$$dS = r\, dr\, d\phi$$
$$\mathbf{n} = \mathbf{k}$$

The corresponding flux integrals are

$$Q_a = \int_{S_a} \mathbf{F}\cdot\mathbf{n} dS = \int_{S_a} \alpha\frac{x^2}{a} + \beta\frac{y^2}{a}\, dS = \int_{S_a}\left(\alpha\frac{x^2}{a^2} + \beta\frac{y^2}{a^2}\right) a\, dS$$
$$= \int_{z=-h}^{z=h}\int_{\phi=-\pi}^{\phi=\pi}\left(\alpha\cos^2\phi + \beta\sin^2\phi\right) a^2\, d\phi\, dz$$
$$= \int_{z=-h}^{z=h} \pi a^2\left(\alpha+\beta\right) dz$$
$$= 2\pi a^2 h\left(\alpha+\beta\right)$$

$$Q_b = \int_{S_b} \mathbf{F}\cdot\mathbf{n} dS = \int_{S_b} -\gamma z\, dS = \int_{S_b} \gamma h\, dS = \gamma h\int_{S_b} dS$$
$$= \pi a^2 h\gamma$$

$$Q_c = \int_{S_c} \mathbf{F}\cdot\mathbf{n} dS = \int_{S_c} \gamma z\, dS = \int_{S_c} \gamma h\, dS = \gamma h\int_{S_c} dS$$
$$= \pi a^2 h\gamma$$

The net flux of nutrient across the surface is

$$Q = \int_S \mathbf{F}\cdot\mathbf{dS} = \int_{S_a} \mathbf{F}\cdot\mathbf{dS} + \int_{S_b} \mathbf{F}\cdot\mathbf{dS} + \int_{S_c} \mathbf{F}\cdot\mathbf{dS}$$
$$= 2\pi a^2 h(a+\beta) + \pi a^2 h\gamma + \pi a^2 h\gamma$$
$$= 2\pi a^2 h(a+\beta+\gamma)$$
$$= V(a+\beta+\gamma)$$

where V is the volume of the cylinder.

Note that while the integral of the vector area vanishes with this being a closed surface, the flux integral need not vanish. [We shall see in the next section that for particular forms of **F** the flux integral will also vanish.]

4.7 The divergence theorem (statement)*

The *divergence theorem* relates the integral of the divergence of a vector field **F** within a volume V to the integral over the bounding surface ∂V of the flux of **F**.

In the example of the nutrient flux across the cylinder (in §4.6.3) we saw that with $\mathbf{F} = \alpha x\mathbf{i} + \beta y\mathbf{j} + \gamma z\mathbf{k}$, the flux was

$$Q = \int_S \mathbf{F}\cdot\mathbf{dS} = V(a+\beta+\gamma) = V\nabla\cdot\mathbf{F} = \int_V \nabla\cdot\mathbf{F}\, dV .$$

* You do not need to understand or be able to use the Divergence Theorem for the A Course.

Indeed, this result generalises to any general differentiable vector field **F**. The *divergence theorem* is thus

$$\int_{\partial V} \mathbf{F} \cdot \mathbf{dS} = \int_{V} \nabla \cdot \mathbf{F} \, dV$$

for a general three-dimensional region (volume) V. As before, ∂V is the surface bounding the volume and **dS** = **n***dS* is the vector area with **n** the outward normal to the surface ∂V.

Proof and use of the divergence theorem is beyond the scope of this course, but we will consider a few brief examples.

Example A

Consider **F**(**x**) = **x**.

Now ∇·**F** = ∇·**x** = 3 (in three dimensions). Hence the divergence theorem predicts

$$\int_{\partial V} \mathbf{F} \cdot \mathbf{dS} = \int_{\partial V} \mathbf{x} \cdot \mathbf{dS} = \int_{V} \nabla \cdot \mathbf{F} \, dV = \int_{V} 3 \, dV = 3V .$$

Example B

One of the four Maxwell equations for electromagnetic fields states that

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \sigma$$

where **E** is the electric field, σ is the charge density (charge per unit volume) and ε_0 is a constant.

Now consider the flux of **E** over a closed surface ∂V enclosing the volume V, and apply the divergence theorem:

$$\int_{\partial V} \mathbf{E} \cdot \mathbf{dS} = \int_{V} \nabla \cdot \mathbf{E} \, dV = \frac{1}{\varepsilon_0} \int_{V} \sigma \, dV .$$

This is *Gauss Theorem*: the flux of the electric field out of a region V is $1/\varepsilon_0$ times the total charge in V.

4.8 Stokes' theorem (statement)*

Consider an open surface S bounded by a closed curve C.

Stokes' theorem states that for a differentiable vector field **F**,

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{dS} = \oint_C \mathbf{F} \cdot d\mathbf{x},$$

i.e. the integral of the flux of the curl of **F** over the surface is equal to the line integral of **F** around the closed curve.

We need to be slightly careful here, to ensure that the direction we integrate around C is consistent with $\mathbf{dS} = \mathbf{n}dS$. In particular, we must compute the integral around C in the anticlockwise direction when looking from the *outside* (remember the normal **n** points towards the *outside*).

Example A

Consider $\mathbf{F} = y^3\mathbf{i} - x^3\mathbf{j} + z^3\mathbf{k}$ and let C be any closed curve that encircles the cylinder $x^2 + y^2 = 1$, $z \le 0$. Compute $\oint_C \mathbf{F} \cdot d\mathbf{x}$.

Now $\nabla\times\mathbf{F} = 0\mathbf{i} + 0\mathbf{j} + (-3x^2 - 3y^2)\mathbf{k} = -3r^2\mathbf{k}$.

On the curved sides of the cylinder $\mathbf{n} = x\mathbf{i} + y\mathbf{j}$, so

$$\mathbf{n}\cdot(\nabla\times \mathbf{F}) = 0,$$

and the only flux of $\nabla\times \mathbf{F}$ in or out of the cylinder is through the ends, and this must be the same through any closed curve encircling the cylinder.

For simplicity, take the circle with $r = 1$ at $z = 0$. The outward normal is $\mathbf{n} = \mathbf{k}$, so

$$\mathbf{n}\cdot(\nabla\times \mathbf{F}) = -3r^2$$

and

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* You do not need to understand or be able to use Stokes' Theorem for the A Course.

$$\oint_C \mathbf{F}\cdot d\mathbf{x} = \int_{z=0,r\le 1} \mathbf{n}\cdot(\nabla\times\mathbf{F})\,dS$$
$$= \int_{\theta=-\pi}^{\theta=\pi}\int_{r=0}^{r=1} -3r^2\, r dr\, d\theta$$
$$= \left[\int_{\theta=-\pi}^{\theta=\pi} d\theta\right]\left[\int_{r=0}^{r=1} -3r^2\, r dr\right]$$
$$= [2\pi]\left[-\frac{3}{4}\right]$$
$$= -\frac{3}{2}\pi$$

Green's theorem

If $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$, then

$$\nabla\times\mathbf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

Let C be a closed curve in the plane $z = 0$, and A be the enclosed area.

Applying Stokes theorem,

$$\oint_C \mathbf{F}\cdot d\mathbf{x} = \oint_C (P dx + Q\, dy)$$
$$= \int_A \mathbf{k}\cdot(\nabla\times\mathbf{F})\,dS = \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

The result

$$\oint_C (P dx + Q\, dy) = \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

is called *Green's Theorem*.

Conservative vector fields

Suppose $\nabla\times\mathbf{F} = 0$ (hence the vector field **F** is conservative and can be expressed as the gradient of a scalar, $\mathbf{F} = \nabla\varphi$).

Consider the integral $\oint_C \mathbf{F}\cdot d\mathbf{x}$ around any closed curve C, and choose a surface S that spans C.

By Stokes' theorem, $\oint_C \mathbf{F}\cdot d\mathbf{x} = \int_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} = 0$.

Hence if $\oint_C \mathbf{F}\cdot d\mathbf{x} = 0$ for all closed curve C, the vector field **F** is conservative, so the integral

$$\int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{F}\cdot d\mathbf{x}$$

is independent of the path chosen between $\mathbf{x}_1$ and $\mathbf{x}_2$, and there is a scalar field φ such that $\mathbf{F} = \nabla\varphi$.