Natural Sciences Tripos Part IA

Mathematical Methods I — Course A

Michaelmas 2023

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"The book of nature cannot be understood unless one first learns to comprehend the language and read the characters in which it is written. It is written in the language of mathematics ... without which it is not humanly possible to understand a single word of it." — **Galileo**

Schedule

Vectors. Vector sum and vector equation of a line. Scalar product, unit vectors, vector equation of a plane. Vector product, vector area, vector and scalar triple products. Orthogonal bases. Cartesian components. Spherical and cylindrical polar coordinates. [5 lectures]

Complex numbers. Complex numbers and complex plane, vector diagrams. Exponential function of a complex variable. $\exp \mathrm{i}\omega t$, complex representations of cos and sin. Hyperbolic functions. [3]

Differential Calculus. Revision for functions of a single variable of differentiation (including differentiation from first principles, product and chain rules) and of stationary values. Elementary curve sketching. Brief mention of the ellipse and its properties. Power series. Statement of Taylor's theorem. Examples to include the binomial expansion, exponential and trigonometric functions, and logarithm. Newton-Raphson method. [5]

Integral calculus. The integral as the limit of a sum. Methods of integration (including by parts and substitution). Examples to include odd and even functions and trigonometric functions. Fundamental theorem of calculus. [3]

Probability. Elementary probability theory. Simple examples of conditional probability. Probability distributions, discrete and continuous, normalisation. Permutations and combinations. Binomial distribution, $(p+q)^n$, binomial coefficients. Normal distribution. Expectation values, mean, variance and its expression in terms of first and second moments. [5]

Examples. Extended examples distributed through the course. [3]

Course websites

https://www.damtp.cam.ac.uk/research/afha/lectures/NST_1A/

https://www.student-systems.admin.cam.ac.uk/moodle
(search for NST Part IA: Mathematics)

Lecture notes and example sheets

Lecture notes and slides will be available online – on the course webpage.

You will be given two examples sheets during Michaelmas term. These sheets will contain a mixture of 'basic skills' questions, and some more challenging questions. Brief solutions will be uploaded to Moodle towards the end of Michaelmas term and at the beginning of Lent term.

Textbooks

The following textbooks are recommended:

- M. L. Boas (1983). *Mathematical Methods in the Physical Sciences*, 3rd edition. Wiley.
- K. F. Riley, M. P. Hobson and S. J. Bence (2006). *Mathematical Methods for Physics and Engineering*, 3rd edition. Cambridge University Press
- E. Kreyszig (2005). *Advanced Engineering Mathematics*, 9th edition. Wiley.
- G. Stephenson (1973). *Mathematical Methods for Science Students*, 2nd edition. Prentice Hall/Pearson.

Student representative

"The Faculty Board of Mathematics asked DAMTP to set up a Staff-Student Committee for Mathematics in the Natural Sciences to provide an opportunity for discussion of matters relating to the courses. The Committee has four staff and three student members, the latter being drawn from the A and B courses in Part IA and from the Part IB course."

Office Hours

- Office Hours: Tues, 10:15am-11:15am, Pav. F, F2.01, Centre for Mathematical Sciences

1 Vectors

Vectors are indispensible in science. Systems in more than one dimension are most conveniently described using the formalism of vectors, while fluid dynamics and electromagnetism are inherently vector theories.

Vectors provide a compact notation that helps facilitate both calculations and physical understanding. Physical laws represented as vector equations are independent of the coordinate system; you can change your point of view (i.e. reference frame) but a vector equation does not change.

1.1 Definitions and properties

- **Scalars** are quantities with magnitude only
  e.g. speed, mass, time, temperature.

- **Vectors** have both magnitude *and* direction

  e.g. displacement, velocity, force, momentum, magnetic field, electric field.

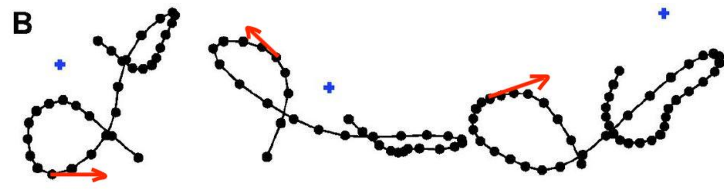


Figure 1: Three actual bee flights recorded by biologists. The instantaneous velocity vector of the bee at each point on the flight is tangent to each curve at that point (indicated by the red arrows).

- Geometrically, vectors are denoted with an arrow. The length of the arrow represents the magnitude of the vector.

- Algebraically, vectors are usually denoted by bold letters, e.g. $\boldsymbol{r}$, and sometimes as underlined letters, e.g. $\underline{r}$ (especially in handwriting). Then the magnitude of the vector is written as $|\boldsymbol{r}|$ or $|\underline{r}|$.

- A vector of unit length is called a *unit vector*. Unit vectors are often denoted with a hat, i.e. $\hat{\boldsymbol{r}}$. And so $|\hat{\boldsymbol{r}}| = 1$.

- Constant vectors are not necessarily tied to a place in space; they are 'moveable' (cf. equipollent). Thus two vectors pointing in the same direction with the same magnitude *are the same*.

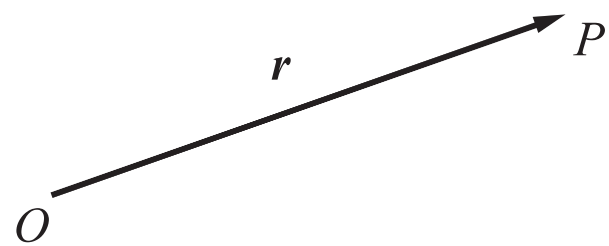


Figure 2: Displacement vector from the origin O to the point P, that is, $\boldsymbol{r} = \overrightarrow{OP}$.

One particular sort of vector is the **displacement** vector between two points, say the displacement of the point P from the point O. It is often denoted by the arrow over-bar notation $\overrightarrow{OP}$. We might also refer to $\overrightarrow{OP}$ as being the displacement vector of the point P *relative to the origin* O, or as P's *position vector*.

1.1.1 Multiplying vectors by scalars

A vector $\boldsymbol{a}$ can be multiplied by a scalar λ, to give another vector $\lambda\boldsymbol{a}$.

- The vector $\lambda\boldsymbol{a}$ is simply the vector $\boldsymbol{a}$ but with its length scaled by λ.
- For λ positive, the direction of the vector remains the same but the magnitude

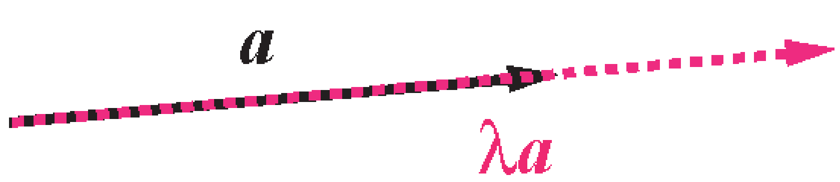


Figure 3: Two vectors: $\boldsymbol{a}$ (solid line) and $\lambda\boldsymbol{a}$ (dashed line)

changes.

- Multiplying by a negative real scalar λ reverses the direction of the vector.

- Two different vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, pointing in the *same* direction must be related by $\boldsymbol{a} = \mu\boldsymbol{b}$, where μ is some scalar.

1.1.2 Adding and subtracting vectors

We can also add and subtract vectors. It is easiest to define this operation geometrically.

- To add $\boldsymbol{a}$ and $\boldsymbol{b}$, arrange them so they form adjacent sides of a triangle, with arrow-tip to tail. Then $\boldsymbol{a} + \boldsymbol{b}$ is simply the third side of the triangle, with its arrow head meeting $\boldsymbol{b}$'s arrow head.

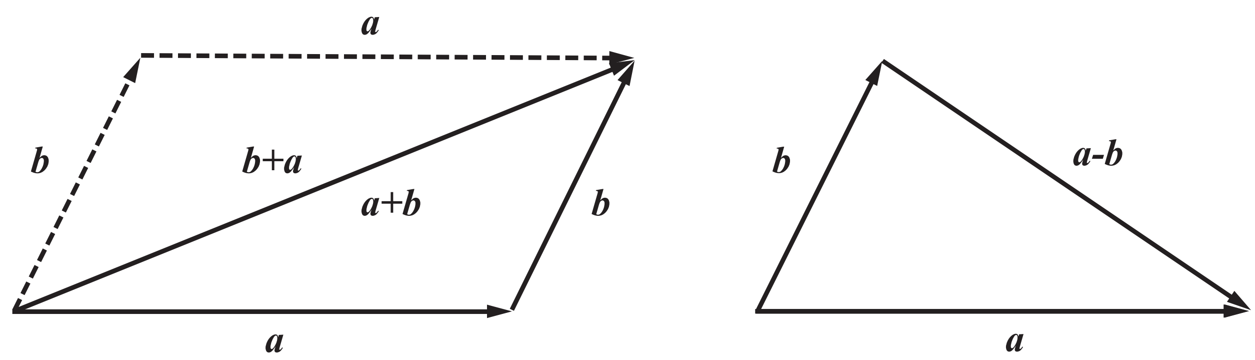


Figure 4: Adding and subtracting two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$.

- If we construct a parallelogram from $\boldsymbol{a}$ and $\boldsymbol{b}$, we see that

$$\boxed{\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}}\,, \tag{1}$$

  i.e. vector addition is **commutative**.

- The difference of two vectors, $\boldsymbol{a}-\boldsymbol{b}$, should be understood as $\boldsymbol{a}+(-\boldsymbol{b})$. To compute it, first multiply $\boldsymbol{b}$ by the scalar -1, then add to $\boldsymbol{a}$ as normal.

- As well as being commutative, vector addition is also **associative**, i.e. the order in which the vectors are added does not matter;

$$\boxed{(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}=\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})}\,. \tag{2}$$

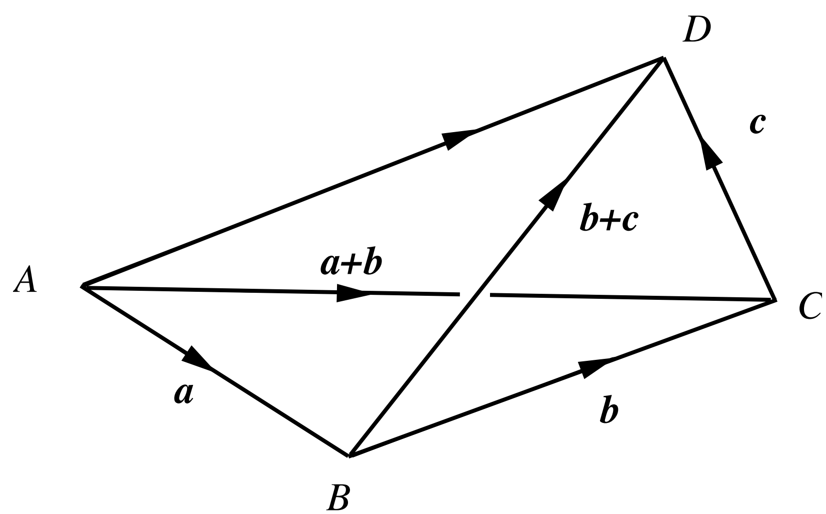


Figure 5: Diagram illustrating the associativity of vector addition.

- Finally, scalar multiplication and vector addition obey the two **distributive** laws; i.e.

$$\boxed{\lambda(\boldsymbol{a} + \boldsymbol{b}) = \lambda\boldsymbol{a} + \lambda\boldsymbol{b}} \tag{3}$$

$$\boxed{(\lambda + \mu)\boldsymbol{a} = \lambda\boldsymbol{a} + \mu\boldsymbol{a}} \tag{4}$$

1.1.3 Cartesian coordinate axes

Vector coordinates in two dimensions - simple case

Consider the point P in the plane, with Cartesian coordinates (x, y).

Next consider the position vector of this point $\boldsymbol{r} = \overrightarrow{OP}$, where O is the origin.

- We can represent such position vectors by the coordinate pair (x, y), and actually write $\boldsymbol{r} = (x, y)$

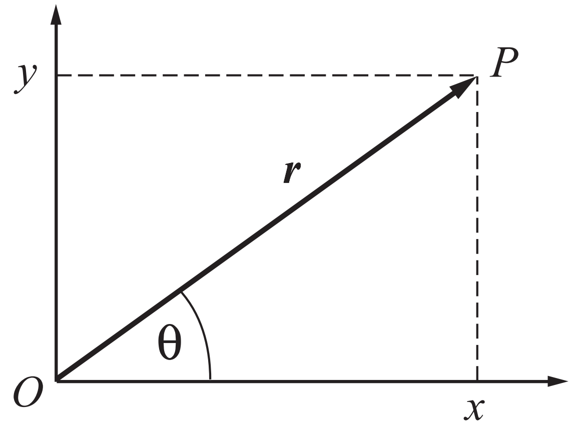


Figure 6: position vector $\boldsymbol{r}$ in the plane with Cartesian axes.

- We introduce two unit vectors, one along the x-axis, denoted by $\hat{\boldsymbol{i}}$ (or sometimes $\hat{\boldsymbol{x}}$), and one along the y-axis, denoted by $\hat{\boldsymbol{j}}$ (or $\hat{\boldsymbol{y}}$). These unit vectors are orthogonal (perpendicular) to each other. They may be represented via coordinates by

$$\hat{\boldsymbol{i}} = (1,\, 0)\,, \qquad \hat{\boldsymbol{j}} = (0, 1)\,. \tag{5}$$

- Via the laws of vector addition, our 2D vector $\overrightarrow{OP}$ can be represented as a *linear combination* of these unit vectors

$$\overrightarrow{OP} = \boldsymbol{r} = x\,\hat{\boldsymbol{i}} + y\,\hat{\boldsymbol{j}}\ , \tag{6}$$

  and we see now that the constant coefficients of $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ are, in fact, also the coordinates of the point P.

- This representation permits us to do vector algebra component-wise: If $\boldsymbol{r}_1 = (x_1, y_1)$ and $\boldsymbol{r}_2 = (x_2, y_2)$ are two position vectors,

$$\lambda\boldsymbol{r_1} = \lambda(x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}}) = (\lambda x_1, \lambda y_1), \tag{7}$$

$$\begin{aligned} \boldsymbol{r}_1 + \boldsymbol{r}_2 = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + (x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}}) &= (x_1 + x_2)\hat{\boldsymbol{i}} + (y_1 + y_2)\hat{\boldsymbol{j}}, \\ &= (x_1 + x_2, y_1 + y_2). \end{aligned} \tag{8}$$

- From Pythagoras' Theorem, the *length* or *magnitude* of the vector $\boldsymbol{r}$ is

$$r = |\boldsymbol{r}| = \sqrt{x^2 + y^2}\ . \tag{9}$$

- The *direction* of the 2D vector $\boldsymbol{r}$ can be represented by the unit vector $\hat{\boldsymbol{r}}$ lying in the same direction, that is,

$$\hat{\boldsymbol{r}} = \left(\frac{x}{r}, \frac{y}{r}\right) = \frac{x}{r}\,\hat{\boldsymbol{i}} + \frac{y}{r}\,\hat{\boldsymbol{j}}\,. \tag{10}$$

  Alternatively we could also represent the direction using the angle θ subtended with the x-axis, where

$$\hat{\boldsymbol{r}} = (\cos\theta,\ \sin\theta) = \cos\theta\,\hat{\boldsymbol{i}} + \sin\theta\,\hat{\boldsymbol{j}}\,. \tag{11}$$

Vector coordinates in three dimensions - general case

- The $coordinates$ of a point P relative to three-dimensional Cartesian axes are denoted x, y, z and correspond to the lengths of the projections of the vector $\overrightarrow{OP}$ onto the three axes.

- As before, we can write $\overrightarrow{OP} = (x, y, z)$.

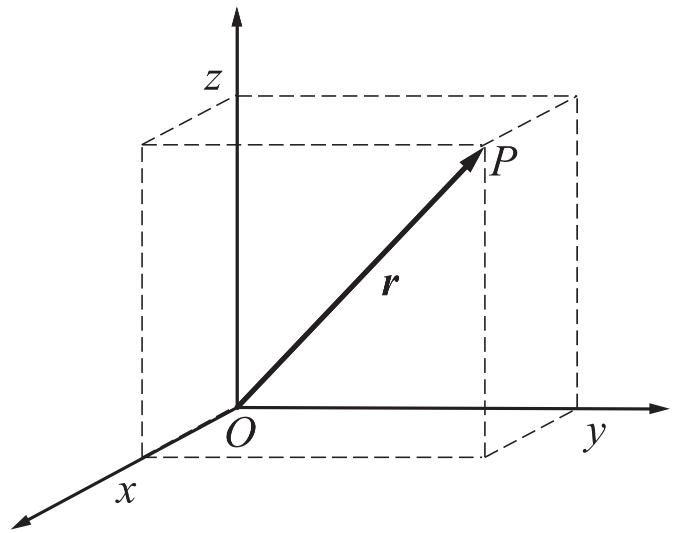


Figure 7: Position vector $\boldsymbol{r}$ in 3D with Cartesian axes.

- We introduce unit vectors $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ which are parallel to the three axes. They are also commonly denoted by $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$, We have that

$$\hat{\boldsymbol{i}} = (1, 0, 0) \qquad \hat{\boldsymbol{j}} = (0, 1, 0) \qquad \hat{\boldsymbol{k}} = (0, 0, 1) \ , \tag{12}$$

and then that

$$\overrightarrow{OP} = \boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}} \ . \tag{13}$$

The last equality follows from vector addition.

- The vectors $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ are said to form a *basis* for three-dimensional space, because any 3D vector can be written as a linear combination of these three vectors.

  Of course, this is not the only basis for three-dimensional space; we could obtain an infinite number of other bases by simply rotating $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ by an arbitrary angle about the origin or by moving (displacing) the origin. We will come back to the idea of bases in section 1.11.

- As in 2D, vector algebra can be done component-wise in 3D. For example,

$$(x_1, y_1, z_1) \pm (x_2, y_2, z_2) = (x_1 \pm x_2, y_1 \pm y_2, z_1 \pm z_2).$$

- The *length* of the vector $\boldsymbol{r}$ can be obtained from Pythagoras' Theorem and is given by

$$\boxed{|\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2}}\ . \tag{14}$$

- The distance between the two points $\boldsymbol{r_1} = (x_1, y_1, z_1)$ and $\boldsymbol{r_2} = (x_2, y_2, z_2)$ can be found by noting first that

$$\boldsymbol{r_1} - \boldsymbol{r_2} = (x_1 - x_2, y_1 - y_2, z_1 - z_2)\ ,$$

so that from (14)

$$|\boldsymbol{r_1} - \boldsymbol{r_2}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}\ . \tag{15}$$

Example 1.1 An aeroplane flies from city A to city B, which is at a distance of 1000km north-west of A. The average speed of the plane relative to the air immediately around it is $800\mathrm{kmh}^{-1}$. For the journey in question, there is a wind of speed $80\mathrm{kmh}^{-1}$ from the south-west. What direction would the pilot need to fly in, and how long does the journey take?

Let $\boldsymbol{v}_{\text{rel}}$ be the velocity of the plane relative to the air, and $\boldsymbol{v}_{\text{gr}}$ its velocity to the ground. We denote by $\overrightarrow{AB}$ the displacement vector between the two cities, and we want to set a course so that $\boldsymbol{v}_{\text{gr}}$ is parallel to $\overrightarrow{AB}$. Finally, $\boldsymbol{v}_{\text{wind}}$ is the wind velocity, perpendicular to $\overrightarrow{AB}$.

The plane's total velocity relative to the ground can be determine from vector addition (the plane propels itself but is also carried by the wind around it) so that

$$\boldsymbol{v}_{\text{gr}} = \boldsymbol{v}_{\text{rel}} + \boldsymbol{v}_{\text{wind}}.$$

The triangle so formed is a right angled triangle, so we can just use trigonometry to get the angle between $\boldsymbol{v}_{\text{rel}}$ and $\boldsymbol{v}_{\text{gr}}$, denoted ϕ:

$$\sin\phi = \frac{|\boldsymbol{v}_{\text{wind}}|}{|\boldsymbol{v}_{\text{rel}}|} = \frac{80}{800} = 0.1,$$

and so $\phi = \sin^{-1}(0.1) = 5.7°$.

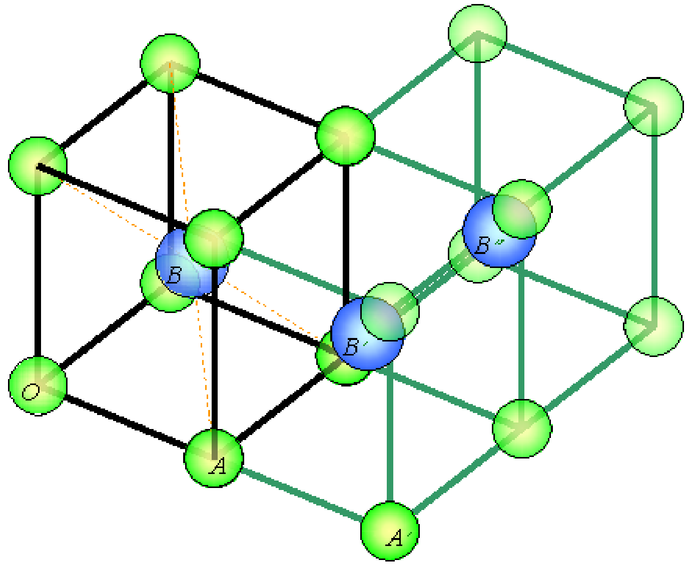
To determine how long the flight takes we need to know $\boldsymbol{v}_{\rm gr}$, which follows from Pythagoras:

$$|\boldsymbol{v}_{\rm gr}| = \sqrt{|\boldsymbol{v}_{\rm rel}|^2 - |\boldsymbol{v}_{\rm wind}|^2} = \sqrt{800^2 - 80^2} = 80 \cdot \sqrt{99} \approx 796.0$$

The flight time is distance divided by speed $= 1000/796 = 1.2563$ hours.

(If there was no wind, the flight time is 1.25 hours, and so the 80 km/h crosswind causes roughly 23 seconds delay!)

Example 1.2 The crystalline lattice of caesium chloride consists of cubic unit cells. Each unit cell consists of a chlorine atom (labelled A) at each corner of the cube, and a caesium atom (labelled B) at the centre of the cube. The sides of the cube have length a. Find the distance from an atom B to its nearest neighbouring atom of type (i) A and (ii) B. What distances are the second nearest neighbours from B?



Let us set up a coordinate system aligned with the lattice and with origin O placed at one of the A atoms. There are hence A atoms at $(0,0,0)$, $(a,0,0)$, $(0,a,0)$. etc. And there are B atoms at $(a/2,a/2,a/2)$, $(3a/2,a/2,a/2)$, $(3a/2,3a/2,a/2)$, etc.

(i) Every B atom has eight A atoms surrounding it and these are equally close to it.

To get this distance choose the A atom at the origin and the B atom at $(a/2, a/2, a/2)$.

The length of the displacement vector is simply

$$|\boldsymbol{r}_A - \boldsymbol{r}_B| = |(0,0,0) - (a/2, a/2, a/2)| = (\sqrt{3}/2)a.$$

(ii) From the diagram the two nearest B atoms are always located along the coordinate axes. Choosing B and B' in the diagram, the distance can be determined from the length of the displacement vector:

$$|\boldsymbol{r}_{B'} - \boldsymbol{r}_B| = \left| \begin{pmatrix} \frac{3}{2}a \\ \frac{1}{2}a \\ \frac{1}{2}a \end{pmatrix} - \begin{pmatrix} \frac{1}{2}a \\ \frac{1}{2}a \\ \frac{1}{2}a \end{pmatrix} \right| = a.$$

There are several A atoms that are second nearest to B. Let us select A' in the figure. Its position vector is $\boldsymbol{r}_{A'} = (2a, 0, 0)$. The distance between the two atoms can be determined from:

$$
\begin{aligned}
|\boldsymbol{r}_{A'} - \boldsymbol{r}_B| &= \left| \begin{pmatrix} 2a \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2}a \\ \frac{1}{2}a \\ \frac{1}{2}a \end{pmatrix} \right| = \left| \begin{pmatrix} \frac{3}{2}a \\ -\frac{1}{2}a \\ -\frac{1}{2}a \end{pmatrix} \right| \\
&= \sqrt{(\tfrac{3}{2}a)^2 + (-\tfrac{1}{2}a)^2 + (-\tfrac{1}{2}a)^2} \\
&= \frac{\sqrt{11}}{2}a.
\end{aligned}
$$

The second-nearest B atom from B we take for our calculation to be B'' with position vector $\boldsymbol{r}_{B''} = (3a/2, 3a/2, a/2)$. We quickly find that

$$|\boldsymbol{r}_{B''} - \boldsymbol{r}_B| = \sqrt{2}a.$$

Example 1.3 Recall that the cosine rule for the triangle illustrated below (Figure 8) is

$$c^2 = a^2 + b^2 - 2ab\cos\gamma\,, \tag{16}$$

where $a = |\boldsymbol{a}| = |\overrightarrow{OA}|$, $b = |\boldsymbol{b}| = |\overrightarrow{OB}|$ and $c = |\boldsymbol{c}| = |\overrightarrow{BA}|$. The sine rule here is

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}\,. \tag{17}$$

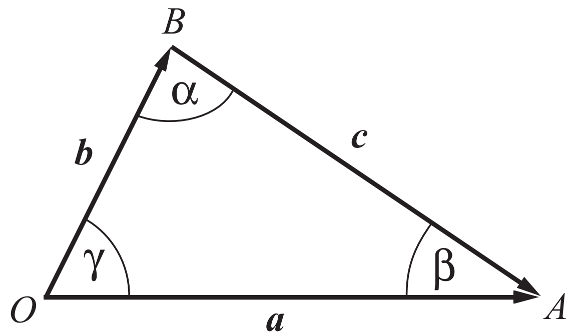


Figure 8: Vector and angle definitions for the cosine and sine rules.

Use the cosine rule to calculate the angle between the lines OA and OB, where O is the origin, $A = (2, -1, 2)$ and $B = (1, 1, 1)$.

We first recognise that the two lines make a triangle once we join the two points A and B.

We next re-arrange the cosine rule to isolate γ the angle between the two lines and the thing we want to find:

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}.$$

Next we need to find all the lengths a, b, and c:

$$\begin{aligned} a &= |\boldsymbol{a}| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3, \\ b &= |\boldsymbol{b}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}, \\ c &= |\boldsymbol{b} - \boldsymbol{a}| = \sqrt{(1-2)^2 + (1+1)^2 + (1-2)^2} = \sqrt{6}, \end{aligned}$$

where we have used vector addition in the last line.

We then just plug the numbers into the formula:

$$\cos\gamma = \frac{9 + 3 - 6}{2 \times 3 \times \sqrt{3}} = \frac{1}{\sqrt{3}},$$

and so $\gamma = \cos^{-1}\sqrt{1/3} = 54.7°$.

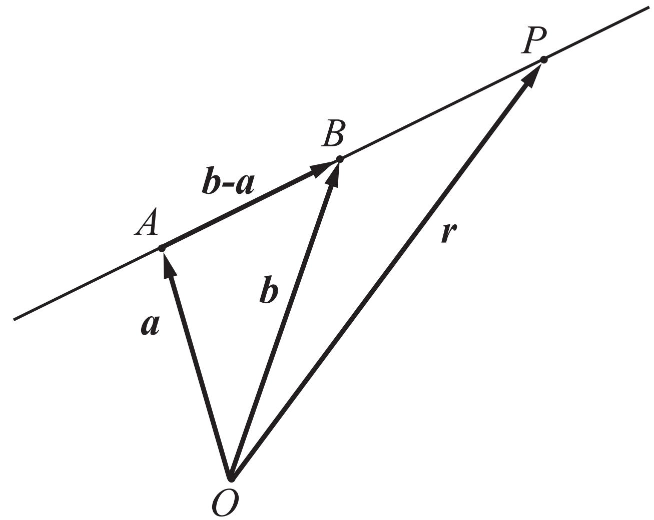


Figure 9: A straight line that goes through points A and B, the origin O, and a random point on the line P.

1.2 The equation of a line

Consider a line in the plane that passes through two points A and B, with $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$. How can we describe a general point P that lies on this line? What equation does its position vector $\boldsymbol{r}$ satisfy?

Note that the line between A and P is necessarily parallel to the line between A and B. This means that $\overrightarrow{AP}$ must be a scalar multiple of $\overrightarrow{AB}$, or in other words

$$\boldsymbol{r} - \boldsymbol{a} = \lambda(\boldsymbol{b} - \boldsymbol{a}) \; ,$$

for some scalar λ. Rearranging yields

$$\boxed{\boldsymbol{r} = \boldsymbol{a} + \lambda(\boldsymbol{b} - \boldsymbol{a})} \; . \tag{18}$$

Equation (18) is the equation of a straight line. Any point $\boldsymbol{r}$ on the line can be written in the form (18) for some λ. The scalar λ runs through all real values.

Instead of knowing two points on the line, we are sometimes given just one point (say $\boldsymbol{a}$) together with the direction of the line (the unit vector $\hat{\boldsymbol{t}}$).

In that case, the equation of the line can easily be modified to give the alternate general (and more convenient) form

$$\boxed{\boldsymbol{r} = \boldsymbol{a} + \lambda \hat{\boldsymbol{t}}}\ . \tag{19}$$

1.2.1 Component description - two dimensions

To keep things simple to start, suppose the line lies in the xy plane.

Now a point on the line can be written as $\boldsymbol{r} = (x, y)$. And the equation of the line, in components, is

$$(x, y) = (a_x + \lambda(b_x - a_x),\ a_y + \lambda(b_y - a_y))\,, \tag{20}$$

where $\boldsymbol{a} = (a_x, a_y)$ and $\boldsymbol{b} = (b_x, b_y)$.

This implies both $\lambda = (x - a_x)/(b_x - a_x)$ and $\lambda = (y - a_y)/(b_y - a_y)$.

Eliminating λ yields the familiar:

$$\frac{x - a_x}{b_x - a_x} = \frac{y - a_y}{b_y - a_y} \qquad \text{or} \qquad y = c\,x + d\,, \tag{21}$$

where the slope $c = (b_y - a_y)/(b_x - a_x)$ and the intercept $d = a_y - c\, a_x$.

1.2.2 Component description - three dimensions

In three dimensions, the component form of the line is a bit more involved. Let now $\boldsymbol{r} = (x, y, z)$. We once again rewrite the equation of a line in Cartesian components:

$$(x, y, z) = (a_x + \lambda(b_x - a_x), a_y + \lambda(b_y - a_y), a_z + \lambda(b_z - a_z)) \ ,$$

where $\boldsymbol{a} = (a_x, a_y, a_z)$ and $\boldsymbol{b} = (b_x, b_y, b_z)$.

Equating the components yields three equations

$$\lambda = \frac{x - a_x}{b_x - a_x} \quad , \ \lambda = \frac{y - a_y}{b_y - a_y} \quad , \quad \lambda = \frac{z - a_z}{b_z - a_z} \ .$$

Of course λ is the same in each of these three equations, so the right hand sides must all be equal to each other.

On eliminating λ, we obtain

$$\boxed{\frac{x - a_x}{b_x - a_x} = \frac{y - a_y}{b_y - a_y} = \frac{z - a_z}{b_z - a_z}}\,, \tag{22}$$

which is an alternative form of (18) not involving the scalar parameter λ.

Example 1.4 Find the equation of the straight line through the points $\boldsymbol{a} = (1, 2, 3)$, $\boldsymbol{b} = (4, 0, 6)$. Does this line pass through the point $(4, 0, 0)$? For what value of μ does the point $(10, -4, \mu)$ lie on the line?

Plug in our information into the line equation:

$$\begin{aligned}\boldsymbol{r} &= \boldsymbol{a} + \lambda(\boldsymbol{b} - \boldsymbol{a}),\\ &= (1, 2, 3) + \lambda[(4, 0, 6) - (1, 2, 3)]\\ &= (1 + 3\lambda, 2 - 2\lambda, 3 + 3\lambda).\end{aligned}$$

Now is $\boldsymbol{r} = (4, 0, 0)$ on this line? Plug it in and see if the line equation can be satisfied for some λ.

The three components of the equation give

$$4 = 1 + 3\lambda, \quad 0 = 2 - 2\lambda, \quad , 0 = 3 + 3\lambda.$$

The first equations yields $\lambda = 1$ but the third equation demands that $\lambda = -1$. So we have a contradiction, which can only be resolved if the point $(4, 0, 0)$ does not lie on the line.

We play the same game with $\boldsymbol{r} = (10, -4, \mu)$. We plug in this expression into the line equation. The x component gives:

$$10 = 1 + 3\lambda \qquad \rightarrow \lambda = 3,$$

while the y component yields $-4 = 2 - 2\lambda$, which is consistent with $\lambda = 3$. Finally the z component is:

$$\mu = 3 + 3\lambda = 12.$$

So the point lies on the line when $\mu = 12$.

Example 1.5 Consider the triangle ABC, with the position vectors of the points relative to the origin O being $\boldsymbol{a}$,$\boldsymbol{b}$,$\boldsymbol{c}$ respectively. Show that the medians (the lines from each vertex to the middle of the opposite side) meet in a single point, the so-called centroid.

Let D be the mid point on the line BC, and let its position vector be $\boldsymbol{d} = \overrightarrow{OD}$. Consider the triangle OBD; from vector addition,

$$\boldsymbol{d} = \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BC} = \boldsymbol{b} + \tfrac{1}{2}(\boldsymbol{c} - \boldsymbol{b}) = \tfrac{1}{2}(\boldsymbol{b} + \boldsymbol{c}).$$

Next let us find the equation of the line joining points A and D , i.e. the median joining the vertex A and the midpoint of BC. This is simply:

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{a} + \lambda(\boldsymbol{d} - \boldsymbol{a}), \\ &= \boldsymbol{a} + \lambda\left[\tfrac{1}{2}(\boldsymbol{b} + \boldsymbol{c}) - \boldsymbol{a}\right], \\ &= (1 - \lambda)\boldsymbol{a} + \tfrac{1}{2}\lambda\boldsymbol{b} + \tfrac{1}{2}\lambda\boldsymbol{c}, \end{aligned}$$

where λ is some real scalar parameter.

If we repeat this process, we can find the equation of the median joining the vertex B and the midpoint of AC:

$$\boldsymbol{r} = (1-\mu)\boldsymbol{b} + \tfrac{1}{2}\mu\boldsymbol{a} + \tfrac{1}{2}\mu\boldsymbol{c},$$

for some other real parameter μ. And indeed the equation of the median joining C to the midpoint of AB is:

$$\boldsymbol{r} = (1-\nu)\boldsymbol{c} + \tfrac{1}{2}\nu\boldsymbol{a} + \tfrac{1}{2}\nu\boldsymbol{b},$$

for a parameter ν.

Now we are in a position to judge where these lines meet and if they do meet at the same point. Let us see where the first two lines intersect, by setting

$$(1-\lambda)\boldsymbol{a} + \tfrac{1}{2}\lambda\boldsymbol{b} + \tfrac{1}{2}\lambda\boldsymbol{c} = (1-\mu)\boldsymbol{b} + \tfrac{1}{2}\mu\boldsymbol{a} + \tfrac{1}{2}\mu\boldsymbol{c}.$$

Equating coefficients of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ gives us three equations. If we look at $\boldsymbol{c}$ first we find $\lambda = \mu$. Then turning to both $\boldsymbol{a}$ and $\boldsymbol{b}$ we get the same equation:

$1 - \lambda = \lambda/2$. And thus

$$\lambda = \mu = \frac{2}{3}.$$

The position vector of the point is then:

$$\boldsymbol{r} = \tfrac{1}{3}\boldsymbol{a} + \tfrac{1}{3}\boldsymbol{b} + \tfrac{1}{3}\boldsymbol{c}.$$

Finally, we need to check that this point lies on the third line. If we plug it in to the equation we have:

$$\tfrac{1}{3}\boldsymbol{a} + \tfrac{1}{3}\boldsymbol{b} + \tfrac{1}{3}\boldsymbol{c} = (1 - \nu)\boldsymbol{c} + \tfrac{1}{2}\nu\boldsymbol{a} + \tfrac{1}{2}\nu\boldsymbol{b},$$

which can be satisfied when $\nu = 2/3$.

So all three medians intersect at the same point.

1.3 Scalar product

We consider two different ways of multiplying vectors together. The first is the **scalar product**, in which two vectors are multiplied together to give a scalar.

Consider the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, whose directions have an angle θ between them (without loss of generality $0 \leqslant \theta \leqslant \pi$).

The *scalar*, or *dot*, product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by the formula

$$\boxed{\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta} \; . \tag{23}$$

Scalar product properties:

1. The scalar product is commutative,

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a} \; , \tag{24}$$

which follows straight from (23).

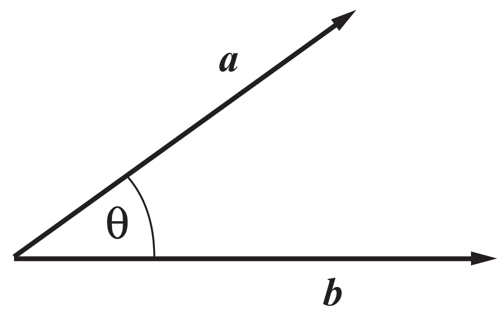


Figure 10: Angle used in the definition of the scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$

2. The scalar product is distributive,

$$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c} \,. \tag{25}$$

3. The scalar product of a vector with itself is denoted by convention as the square of the vector, i.e. $\boldsymbol{a}^2 \equiv \boldsymbol{a} \cdot \boldsymbol{a}$. It follows from (23) that

$$\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2 \,,$$

i.e. the length of $\boldsymbol{a}$ squared. Using (25) and then (24) yields the useful result

$$(\boldsymbol{a}+\boldsymbol{b})^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 + 2\boldsymbol{a}\cdot\boldsymbol{b}\ . \tag{26}$$

4. If $\boldsymbol{a}\cdot\boldsymbol{b} = 0$, and $|\boldsymbol{a}|, |\boldsymbol{b}| \neq 0$ then $\cos\theta = 0$, necessarily, i.e. the vectors are orthogonal.

5. The unit vectors $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ have the following properties:

$$\begin{array}{rcccccl} \hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{i}} &=& \hat{\boldsymbol{j}}\cdot\hat{\boldsymbol{j}} &=& \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}} &=& 1\,, \\ \hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{j}} &=& \hat{\boldsymbol{j}}\cdot\hat{\boldsymbol{k}} &=& \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{i}} &=& 0 \end{array}\ . \tag{27}$$

   In fact, $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ are often referred to as being an *orthonormal* basis set; 'ortho-' meaning orthogonal to each other and 'normal' meaning that each has length unity.

1.3.1 Component formula

We now derive an expression for the scalar product of two vectors $\boldsymbol{a} = (a_x, a_y, a_z)$ and $\boldsymbol{b} = (b_x, b_y, b_z)$ which are given in component form. We have

$$\boldsymbol{a} \cdot \boldsymbol{b} = (a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}} + a_z\hat{\boldsymbol{k}}) \cdot (b_x\hat{\boldsymbol{\imath}} + b_y\hat{\boldsymbol{\jmath}} + b_z\hat{\boldsymbol{k}}) \ .$$

We use the distributive property (25) to expand the brackets, and then the commutative property (24) to see that $\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\imath}}$ etc. This produces

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x \hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}} \ &+ \ a_y b_y \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}} + a_z b_z \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} + (a_x b_y + a_y b_x) \hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}} \\ &+ \ (a_y b_z + a_z b_y) \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{k}} + (a_z b_x + a_x b_z) \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{\imath}} \ , \end{aligned}$$

and finally using the orthonormality in (27) gives us

$$\boxed{\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z} \ . \tag{28}$$

Exercise: Use the property of the scalar product (26) to prove the cosine rule (16). [Hint: See the triangle in Figure 8 with $\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b}$.]

1.3.2 Component and projection of a vector in a given direction

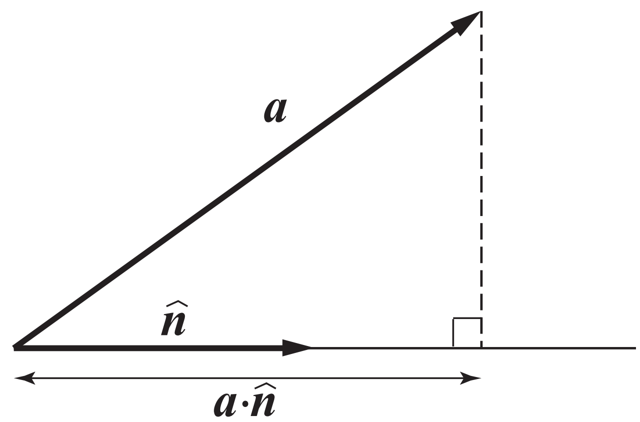


Figure 11: The component of a vector $\boldsymbol{a}$ in the direction of the unit vector $\hat{\boldsymbol{n}}$.

Often we need to work out the component of a vector $\boldsymbol{a}$ in a given direction, say in the direction of the unit vector $\hat{\boldsymbol{n}}$.

By simple trigonometry, this component is $|\boldsymbol{a}|\cos\theta$, where θ is the angle between $\boldsymbol{a}$ and $\hat{\boldsymbol{n}}$.

But this can be rewritten using the scalar product: the component of the vector $\boldsymbol{a}$ in the direction of the unit vector $\hat{\boldsymbol{n}}$ is just $\boldsymbol{a} \cdot \hat{\boldsymbol{n}}$.

The **projection** of $\boldsymbol{a}$ in the direction $\hat{\boldsymbol{n}}$ is a vector and is given by $(\boldsymbol{a} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}}$. It is sometimes also referred to as the 'vector resolution' in $\hat{\boldsymbol{n}}$.

Example 1.6 Find the angle between the vectors $(1, 2, 2)$ and $(0, 3, -4)$. Find the component of the vector $(2, -1, 7)$ in the direction $(1, 1, 1)$.

Call the first vector $\boldsymbol{a}$ and the second $\boldsymbol{b}$. To get the angle θ between them use the dot product.

We know using coordinates that

$$\boldsymbol{a} \cdot \boldsymbol{b} = (1, 2, 2) \cdot (0, 3, -4) = 1 \times 0 + 2 \times 3 + 2 \times (-4) = -2.$$

But it also equals $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$. The amplitudes are easy enough to work out

$$|\boldsymbol{a}| = \sqrt{1 + 2^2 + 2^2} = \sqrt{9} = 3, \quad |\boldsymbol{b}| = \sqrt{0 + 3^2 + (-4)^2} = 5.$$

We then equate the two expressions for the dot product and solve for θ:

$$-2 = 15\cos\theta, \qquad \rightarrow \theta = \cos^{-1}\left(\frac{-2}{15}\right) = 97.7^\circ.$$

The component of the vector $\boldsymbol{c} = (2, -1, 7)$ in the direction $\boldsymbol{n} = (1, 1, 1)$ is just $\boldsymbol{c} \cdot \hat{\boldsymbol{n}}$. The magnitude of $\boldsymbol{n}$ is just $\sqrt{3}$, so

$$\boldsymbol{c} \cdot \hat{\boldsymbol{n}} = (2, -1, 7) \cdot \frac{1}{\sqrt{3}}(1, 1, 1) = \frac{8}{\sqrt{3}}.$$

1.4 The equation of a plane

A general plane in three dimensions can be specified by

1. The orientation of the plane, given by the direction of the unit normal vector to the plane $\hat{\boldsymbol{n}}$

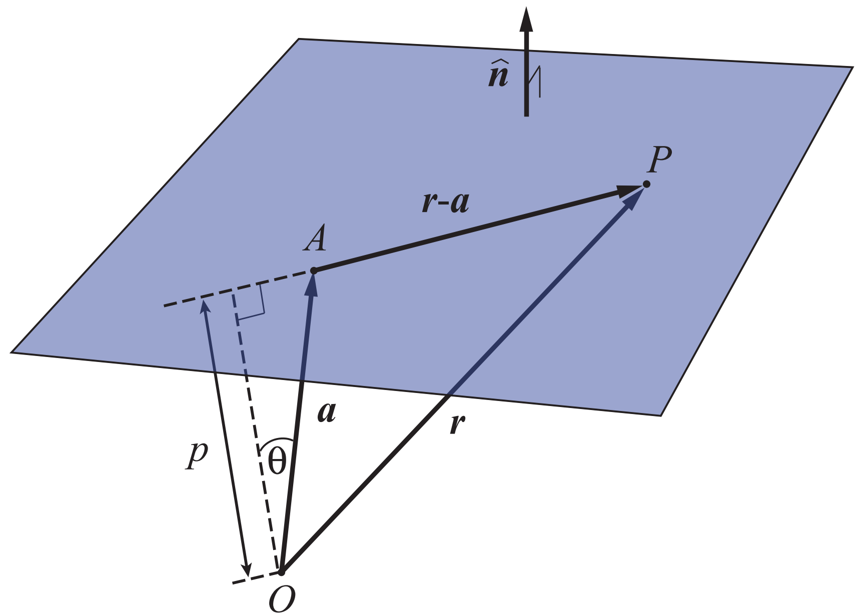


Figure 12: A plane defined by the unit normal vector $\hat{\boldsymbol{n}}$ and the point A.

2. One point on the plane, with position vector $\boldsymbol{a}$, say.

Take the general point P on the plane to have position vector $\boldsymbol{r}$. The vector $\boldsymbol{r} - \boldsymbol{a}$ must lie in the plane, and thus is perpendicular to the normal to the plane

$\hat{\boldsymbol{n}}$, and therefore by property (4) of the scalar product

$$\boxed{(\boldsymbol{r} - \boldsymbol{a}) \cdot \hat{\boldsymbol{n}} = 0}\ , \tag{29}$$

which is one form of the general equation of a plane.

One important property of the plane is its perpendicular distance p to the origin O. From trigonometry

$$p = |\boldsymbol{a}| \cos\theta\ ,$$

but since θ is the angle between $\boldsymbol{a}$ and $\hat{\boldsymbol{n}}$ we have that $p = \boldsymbol{a} \cdot \hat{\boldsymbol{n}}$.
In fact, this yields another form of the equation of the plane,

$$\boxed{\boldsymbol{r} \cdot \hat{\boldsymbol{n}} = p}\ . \tag{30}$$

1.4.1 Component formula

We can also write the equation of the plane in component form. Suppose that $\boldsymbol{r} = (x, y, z)$ and $\hat{\boldsymbol{n}} = (l, m, n)$, then the equation of the plane is simply

$$\boxed{lx + my + nz = p}\ . \tag{31}$$

The real numbers l, m, n are called the *direction cosines*, because they correspond to the cosine of the angle between the normal $\hat{\boldsymbol{n}}$ and the x, y, z axes respectively.

For instance $l = \hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{n}} = \cos\vartheta$, where ϑ is the angle between $\hat{\boldsymbol{n}}$ and the x-axis, i.e. $\hat{\boldsymbol{i}}$. Similarly, m and n are the cosines of the angles made by $\hat{\boldsymbol{n}}$ with $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ respectively.

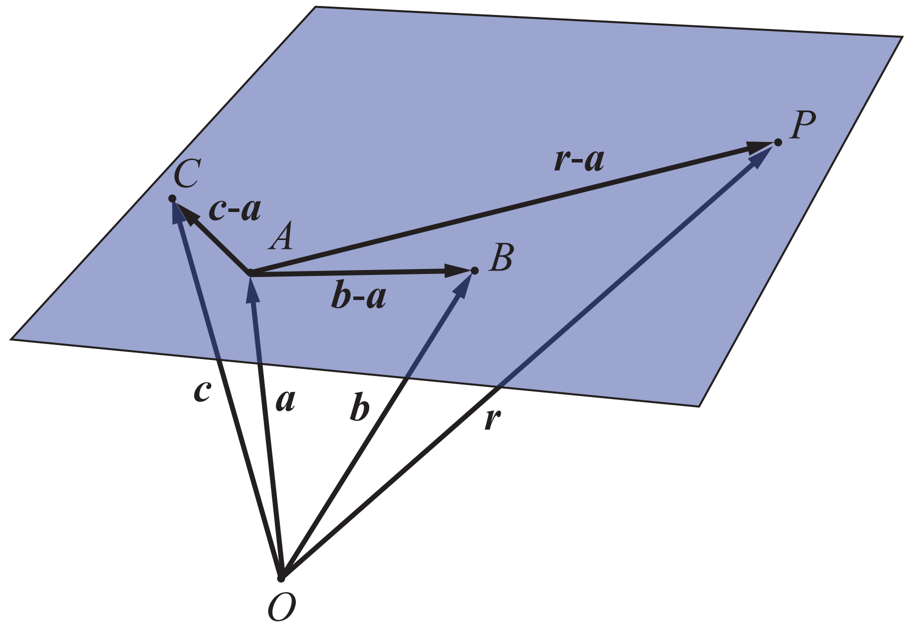


Figure 13: A plane containing the points $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and the general point $\boldsymbol{r}$.

1.4.2 Alternative equation of the plane (if given three points)

We can also find the equation of the plane if we know three points in the plane.

- Suppose that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are the position vectors of three points A, B and C

in the plane.

- Then the vectors $\boldsymbol{c}-\boldsymbol{a}$ and $\boldsymbol{b}-\boldsymbol{a}$ both lie entirely within the plane.

- The plane itself is two-dimensional, and that means that any other vector which lies in the plane can be made as a linear combination of $\boldsymbol{c}-\boldsymbol{a}$ and $\boldsymbol{b}-\boldsymbol{a}$.

- In particular, the vector $\boldsymbol{r}-\boldsymbol{a}$ lies within the plane, so $\boldsymbol{r}-\boldsymbol{a}$ must be the sum of a scalar multiple of $\boldsymbol{b}-\boldsymbol{a}$ plus a scalar multiple of $\boldsymbol{c}-\boldsymbol{a}$, i.e.

$$\boxed{\boldsymbol{r}-\boldsymbol{a}=\mu(\boldsymbol{b}-\boldsymbol{a})+\nu(\boldsymbol{c}-\boldsymbol{a})}\ . \tag{32}$$

Equation (32) is yet another form for the equation of a plane, and the idea is that the scalars μ, ν range over all real values so as to describe all possible points on the plane; we clearly need two degrees of freedom, μ, ν, to describe a two-dimensional object.

Note that in deriving (32) we have assumed that $\boldsymbol{c}-\boldsymbol{a}$ and $\boldsymbol{b}-\boldsymbol{a}$ are *not* parallel to each other.

Example 1.7 Find the line of intersection of the two planes with normals parallel to the vectors $(1, 0, 2)$ and $(-1, 1, 1)$ which both pass through the point $(1, 1, 0)$. What angles does this line make with the coordinate axes?
The equation of the plane may be written as $(\boldsymbol{r}-\boldsymbol{a})\cdot\boldsymbol{n} = 0$ where $\boldsymbol{a}$ is a point on the plane and $\boldsymbol{n}$ is a vector perpendicular to the plane. Note that $\boldsymbol{n}$ need not be a unit vector for this equation to hold (though it is sometimes convenient to frame things this way).
The equation of the first plane can then be written:

$$[(x, y, z) - (1, 1, 0)] \cdot (1, 0, 2) = 0, \qquad \rightarrow \qquad x - 1 + 2z = 0.$$

The equation of the second plane is:

$$[(x, y, z) - (1, 1, 0)] \cdot (-1, 1, 1) = 0, \qquad \rightarrow \qquad x - y - z = 0.$$

Intersection happens on points that satisfy both equations. We deduct the second equation from the first and get $y + 3z = 1$, and so $z = (1 - y)/3$, putting this in to the first plane's equation yields $z = (1 - x)/2$. Thus the equations of the line may be written

$$\frac{x-1}{-2} = \frac{y-1}{-3} = z.$$

Comparing with Eq.(22) a vector parallel to the line can be determined by looking at the denominators. Call it $\boldsymbol{t} = (-2, -3, 1)$. Its magnitude is $\sqrt{4+9+1} = \sqrt{14}$. Therefore

$$\hat{\boldsymbol{t}} = \left(\frac{-2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right).$$

However the components of $\hat{\boldsymbol{t}}$ are also the direction cosines with respect to three coordinate axes, i.e. $\cos\theta_x$, $\cos\theta_y$, $\cos\theta_z$. So it's easy to compute these angles:

$$\theta_x = \cos^{-1}\left(\frac{-2}{\sqrt{14}}\right) = 122^\circ,$$
$$\theta_y = \cos^{-1}\left(\frac{-3}{\sqrt{14}}\right) = 143^\circ,$$
$$\theta_z = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) = 74^\circ.$$

Example 1.8 Can an object at the point $(-2, 3, 3/2)$, be seen by an observer B at $(4, 5, 1)$ when there is an intervening wall with top given by the line

$$\boldsymbol{r} = (1, 2, 1) + \lambda(1, -1, -1)$$

[the z axis is the upward vertical]?

The observer and the line along the top of the wall define a plane in 3D space. Objects above this plane (with respect to z) can be seen by the observer and objects below this plane cannot be seen. Our job is to determine the equation of the plane and then its height z at the xy location of the object.

Equation of the plane. We know two points on the plane A, given by $\boldsymbol{a} = (1, 2, 1)$ and B. We can find a third point C by going to the line in the plane and setting $\lambda = 1$ which gives $\boldsymbol{c} = \boldsymbol{a} + (1, -1, 1)\,[= (2, 1, 0)]$. We can use the equation of the plane with these three points:

$$\boldsymbol{r} - \boldsymbol{a} = \mu(\boldsymbol{b} - \boldsymbol{a}) + \nu(\boldsymbol{c} - \boldsymbol{a}),$$

where μ and ν are real scalar parameters. We thus have

$$\boldsymbol{r} = (1, 2, 1) + \mu(3, 3, 0) + \nu(1, -1, -1).$$

Equating components we have

$$x = 1 + 3\mu + \nu, \quad y = 2 + 3\mu - \nu, \quad z = 1 - \nu.$$

Now, the object is at $(-2, 3, 3/2)$. Putting in the horizontal position $(x = -2, y = 3)$ in to the equation of the plane yields two equations for μ and ν:

$$-2 = 1 + 3\mu + \nu, \qquad 3 = 2 + 3\mu - \nu,$$

which we solve simultaneously. Add the two equations together and we get $1 = 3 + 6\mu$, thus $\mu = -1/3$. And then $\nu = -2$. This means at this horizontal location the height of the plane is

$$z = 1 - \nu = 3,$$

which is *above* the height of the object $(z = 3/2)$, and so the object *cannot* be viewed by the observer!

1.5 Equations of other objects

1.5.1 Sphere

Consider a sphere of radius R and centre whose position vector is $\boldsymbol{a}$.

A point $\boldsymbol{r}$ on the surface of the sphere is, by definition, a distance R from the centre $\boldsymbol{a}$, and it follows that the equation of the sphere is

$$\boxed{|\boldsymbol{r} - \boldsymbol{a}| = R} \,. \tag{33}$$

In component form with $\boldsymbol{a} = (a_x,\, a_y,\, a_z)$, we obtain the more familiar form of the equation for the sphere

$$(x - a_x)^2 + (y - a_y)^2 + (z - a_z)^2 = R^2 \,.$$

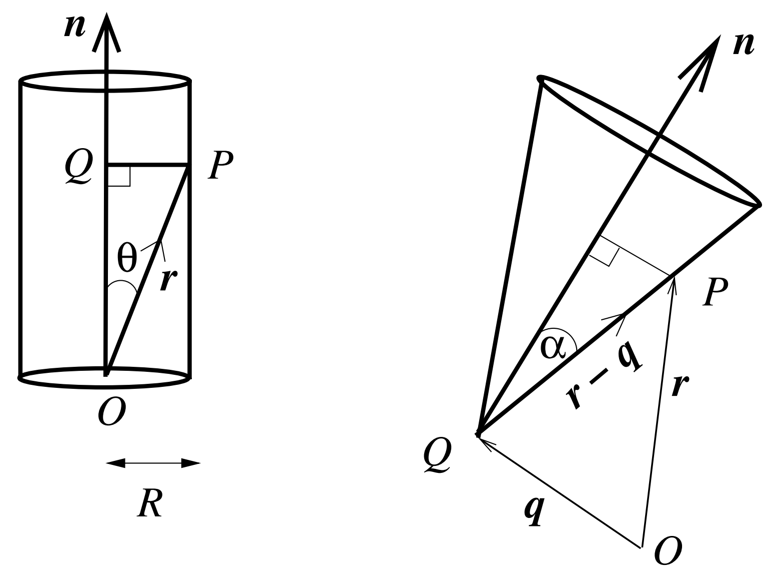


Figure 14: A cylinder and a cone.

1.5.2 Cylinder

Consider a circular cylinder of radius R with axis parallel to the unit vector $\hat{\boldsymbol{n}}$. The origin O lies on the cylinder axis.

- A general point on the surface P has position vector $\overrightarrow{OP} = \boldsymbol{r}$.

- Suppose $\overrightarrow{OQ}$ is the projection of $\overrightarrow{OP}$ onto the cylinder's axis. The distance between O and Q is

$$|\overrightarrow{OQ}| = |\boldsymbol{r}|\cos\theta = |\boldsymbol{r}\cdot\hat{\boldsymbol{n}}| \tag{34}$$

  Since $\overrightarrow{OQ}$ is parallel to the cylinder axis,

$$\overrightarrow{OQ} = (\boldsymbol{r}\cdot\hat{\boldsymbol{n}})\hat{\boldsymbol{n}}.$$

- We can also see that $\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$, and that $|\overrightarrow{QP}| = R$ the cylinder radius. Putting all this information together leads us to the vector equation of the cylinder

$$\boxed{|\boldsymbol{r} - (\boldsymbol{r}\cdot\hat{\boldsymbol{n}})\hat{\boldsymbol{n}}| = R}\ . \tag{35}$$

If the cylinder lies along the z-direction with $\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}}$, then in components the equation is simply $x^2 + y^2 = R^2$.

1.5.3 Cone

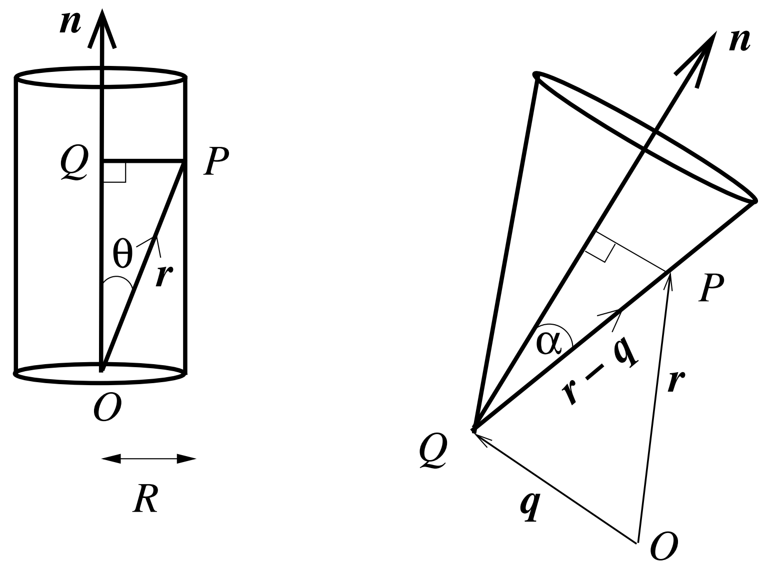


Figure 15: A cylinder and a cone.

Consider a cone with semi-angle α, its axis parallel to the unit vector $\hat{\boldsymbol{n}}$, and its vertex at the point Q with position vector $\boldsymbol{q}$.

- A general point on the cone surface is P with $\overrightarrow{OP} = \boldsymbol{r}$.

- The defining property of the cone is that the line from the vertex to any point on the surface always makes an angle α with axis. Mathematically this means

$$\boxed{(\boldsymbol{r}-\boldsymbol{q})\cdot\hat{\boldsymbol{n}} = |\boldsymbol{r}-\boldsymbol{q}|\cos\alpha}\,, \tag{36}$$

  which is the vector equation of the cone.

1.6 Vector product

We now introduce the *vector product* of two vectors, $\boldsymbol{a}\wedge\boldsymbol{b}$, which is itself a vector. This is also sometimes referred to as the *cross product*, and is sometimes given the notation $\boldsymbol{a}\times\boldsymbol{b}$.

The definition of the vector product is

$$\boxed{\boldsymbol{a}\wedge\boldsymbol{b} \equiv |\boldsymbol{a}||\boldsymbol{b}|\sin\theta\ \hat{\boldsymbol{n}}}\,, \tag{37}$$

where θ is the angle between the two vectors with $0\leqslant\theta\leqslant\pi$ and $\hat{\boldsymbol{n}}$ is a unit

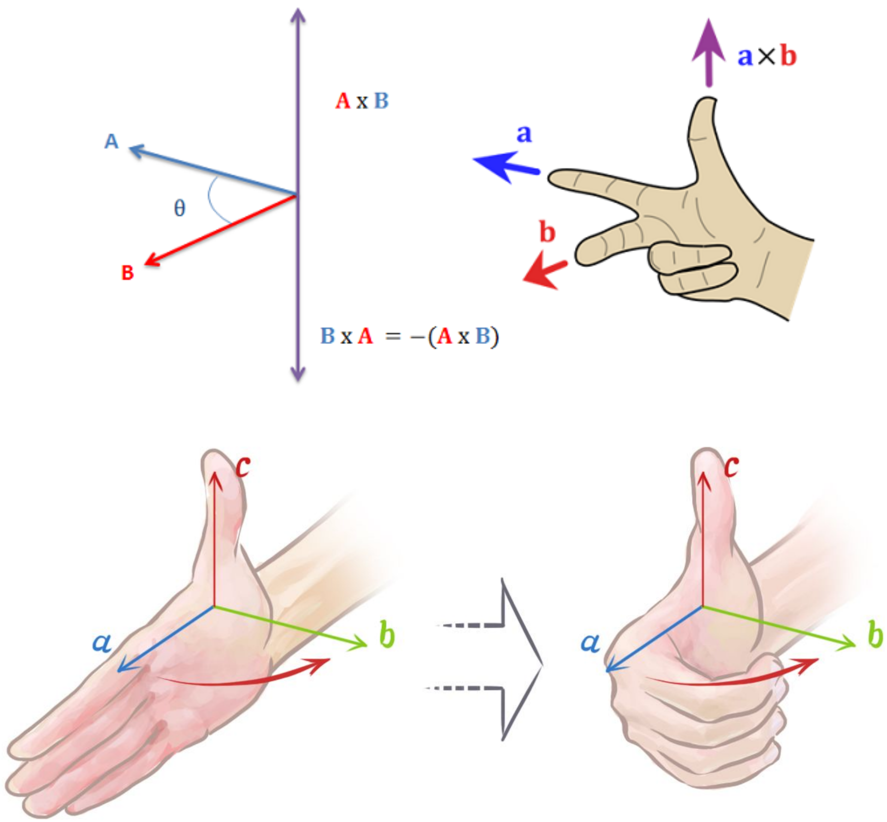


Figure 16: Ways to think about the right hand rule. In the second diagram $\boldsymbol{c} = \boldsymbol{a} \wedge \boldsymbol{b}$.

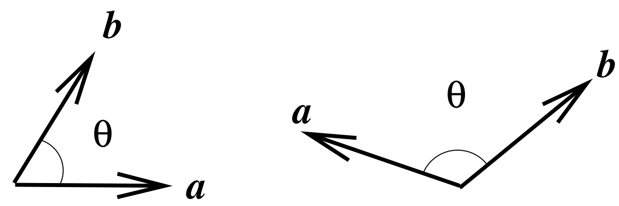


Figure 17: In the example on the left $\boldsymbol{a} \wedge \boldsymbol{b}$ points ***out*** of the page, while in the example on the right $\boldsymbol{a} \wedge \boldsymbol{b}$ points ***into*** the page.

vector. The direction of $\boldsymbol{a} \wedge \boldsymbol{b}$, i.e. $\hat{\boldsymbol{n}}$, is the direction which is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ in such a way that the set $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a} \wedge \boldsymbol{b}$ forms a **right-handed** system.

Note the following properties of $\boldsymbol{a} \wedge \boldsymbol{b}$:

1. The operation is *anti-commutative*,

$$\boldsymbol{a} \wedge \boldsymbol{b} = -\boldsymbol{b} \wedge \boldsymbol{a} \ , \tag{38}$$

which simply follows from the fact that the direction of the cross product reverses when the order is swapped, so as to keep a right-handed system.

2. The cross-product is distributive,

$$\boldsymbol{a} \wedge (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \wedge \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{c} \ . \tag{39}$$

3. The cross-product is *not* associative,

$$\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c}) \neq (\boldsymbol{a} \wedge \boldsymbol{b}) \wedge \boldsymbol{c} \ , \tag{40}$$

   we will establish this in section 1.9.

4. The cross product of two parallel vectors is zero, since then $\theta = 0$ in (37). Thus

$$\hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{i}} = \hat{\boldsymbol{j}} \wedge \hat{\boldsymbol{j}} = \hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{k}} = \boldsymbol{0} \ . \tag{41}$$

5. The vectors $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ are related via

$$\begin{aligned} \hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{j}} &= \hat{\boldsymbol{k}}\,, \\ \hat{\boldsymbol{j}} \wedge \hat{\boldsymbol{k}} &= \hat{\boldsymbol{i}}\,, \\ \hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{i}}. &= \hat{\boldsymbol{j}}\,. \end{aligned} \tag{42}$$

These results can be checked simply using (37), noting that $\theta = \pi/2$ in each case and noting right-handedness (contrast e.g. $\hat{\boldsymbol{j}} \wedge \hat{\boldsymbol{i}} = -\hat{\boldsymbol{k}}$).

1.6.1 Component formula

We can now evaluate the general vector product $\boldsymbol{a} \wedge \boldsymbol{b}$ in components. We have

$$\boldsymbol{a} \wedge \boldsymbol{b} = (a_x\hat{\boldsymbol{i}} + a_y\hat{\boldsymbol{j}} + a_z\hat{\boldsymbol{k}}) \wedge (b_x\hat{\boldsymbol{i}} + b_y\hat{\boldsymbol{j}} + b_z\hat{\boldsymbol{k}})\ .$$

The brackets can be expanded using the distributive property (39):

$$\boldsymbol{a} \wedge \boldsymbol{b} \ = \ a_xb_x\hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{i}} + a_xb_y\hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{j}} + a_xb_z\hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{k}}$$

$$
\begin{aligned}
&+a_y b_x \hat{\boldsymbol{j}} \wedge \hat{\boldsymbol{i}} + a_y b_y \hat{\boldsymbol{j}} \wedge \hat{\boldsymbol{j}} + a_y b_z \hat{\boldsymbol{j}} \wedge \hat{\boldsymbol{k}} \\
&+a_z b_x \hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{i}} + a_z b_y \hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{j}} + a_z b_z \hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{k}} \ . \qquad (43)
\end{aligned}
$$

We then remove all terms involving cross products of vectors with themselves using (41):

$$\boldsymbol{a} \wedge \boldsymbol{b} = a_x b_y \hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{j}} + a_x b_z \hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{k}} + a_y b_x \hat{\boldsymbol{j}} \wedge \hat{\boldsymbol{i}} + a_y b_z \hat{\boldsymbol{j}} \wedge \hat{\boldsymbol{k}} + a_z b_x \hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{i}} + a_z b_y \hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{j}} \ . \qquad (44)$$

We can now employ (38) to collect terms in pairs (e.g. the one involving $\hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{j}}$ together with the one involving $\hat{\boldsymbol{j}} \wedge \hat{\boldsymbol{i}}$):

$$\boldsymbol{a} \wedge \boldsymbol{b} = (a_y b_z - a_z b_y) \hat{\boldsymbol{j}} \wedge \hat{\boldsymbol{k}} + (a_z b_x - a_x b_z) \hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{i}} + (a_x b_y - a_y b_x) \hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{j}} \ . \qquad (45)$$

Finally, we use (42) to obtain the component form of the vector product,

$$\boxed{\boldsymbol{a} \wedge \boldsymbol{b} = (a_y b_z - a_z b_y) \hat{\boldsymbol{i}} + (a_z b_x - a_x b_z) \hat{\boldsymbol{j}} + (a_x b_y - a_y b_x) \hat{\boldsymbol{k}}} \ . \qquad (46)$$

The formula (46) is quite cumbersome and hard to remember, but can be expressed much more compactly using the notation of *determinants* (see later).

We write $\boldsymbol{a} \wedge \boldsymbol{b}$ as

$$\boldsymbol{a} \wedge \boldsymbol{b} \equiv \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}, \tag{47}$$

and the right-hand side is expanded using the following pattern or rules.

A second order determinant expands as:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \,. \tag{48}$$

The third order determinant, which defines the vector product, then expands as:

$$\boldsymbol{a} \wedge \boldsymbol{b} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \hat{\boldsymbol{i}} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \hat{\boldsymbol{j}} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \hat{\boldsymbol{k}} \,. \tag{49}$$

It is well worth remembering the cyclic pattern of indices in this form.

1.6.2 Area of a triangle

Consider a triangle with vertices A, B, and C. The 'side-angle-side' formula for its area is $\frac{1}{2}|AC||AB|\sin\theta$, where θ is the angle made at vertex A. This formula can be rewritten using the definition of the vector product. The area is then $\frac{1}{2}|\overrightarrow{AC} \times \overrightarrow{AB}|$.

Exercise: Use the definition of the vector product (39) to prove the sine rule (17). [Hint: For the triangle in Figure 8, note that $\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b}$ and take the vector product of both sides with $\boldsymbol{a}$.]

1.6.3 Equations for a line and plane (revisited)

We can use the vector product to derive alternative equations for a line and plane.

For instance, take the straight line we considered in Section 1.2.

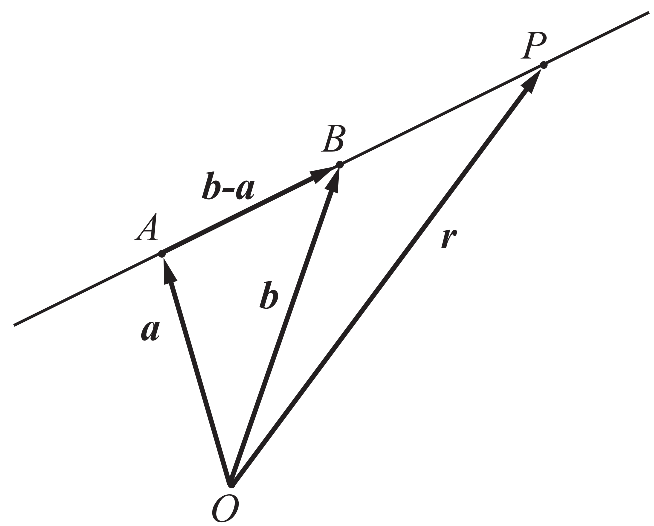


Figure 18: A straight line that goes through points A and B, the origin O, and a random point on the line P. Recall that $\boldsymbol{r} = \boldsymbol{a} + \lambda(\boldsymbol{b} - \boldsymbol{a})$

As we can see, the vector $\boldsymbol{r} - \boldsymbol{a}$ is **parallel** to the vector $\boldsymbol{b} - \boldsymbol{a}$ for any point $\boldsymbol{r}$ on the line. Thus

$$(\boldsymbol{r} - \boldsymbol{a}) \wedge (\boldsymbol{b} - \boldsymbol{a}) = 0 \ . \tag{50}$$

This is another form for the equation of a line through the points $\boldsymbol{a}$ and $\boldsymbol{b}$.

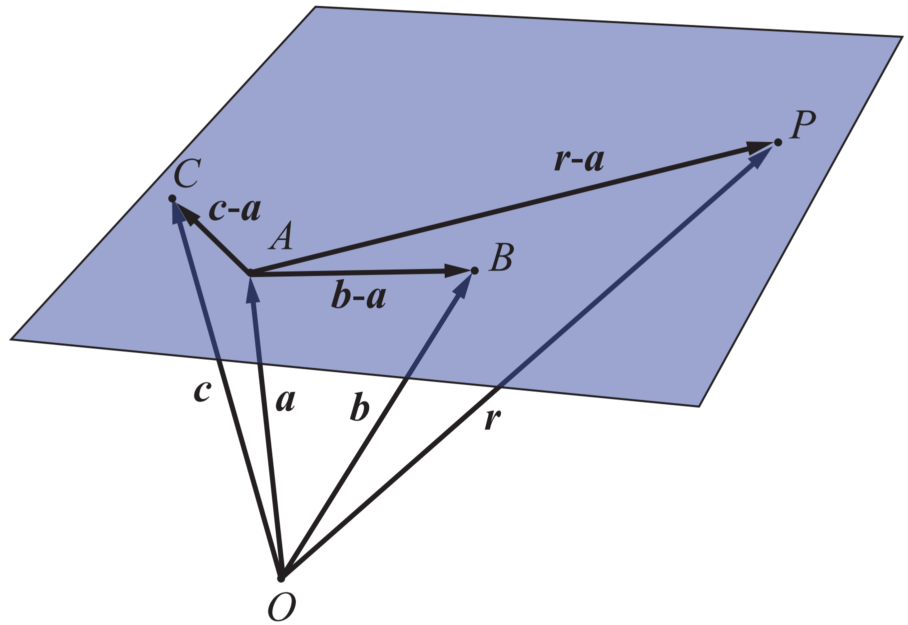


Figure 19: A plane containing the points $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and the general point $\boldsymbol{r}$. Recall the equation for a plane $\boldsymbol{r} - \boldsymbol{a} = \mu(\boldsymbol{b} - \boldsymbol{a}) + \nu(\boldsymbol{c} - \boldsymbol{a})$

For the equation of a plane, suppose we know three points in the plane (A, B, C) as earlier.

- The vectors $\boldsymbol{b} - \boldsymbol{a}$ and $\boldsymbol{c} - \boldsymbol{a}$ lie in the plane.

- By the definition of the vector product, $(\boldsymbol{b}-\boldsymbol{a}) \wedge (\boldsymbol{c}-\boldsymbol{a})$ is the normal (but not necessarily the unit normal) to the plane.

- For any point $\boldsymbol{r}$ in the plane, the vector $\boldsymbol{r}-\boldsymbol{a}$ lies in the plane, and is therefore perpendicular to the normal to the plane. Using property 4 of the scalar product, it therefore follows that

$$(\boldsymbol{r}-\boldsymbol{a}) \cdot [(\boldsymbol{b}-\boldsymbol{a}) \wedge (\boldsymbol{c}-\boldsymbol{a})] = 0 \tag{51}$$

  which is another form for the equation of the plane passing through the points $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

Example 1.9 Determine a unit vector that is perpendicular to both the vectors $(2, 1, 0)$ and $(3, 0, 1)$. Use the cross product to find the angle between these vectors

Call the first vector $\boldsymbol{a}$ and the second $\boldsymbol{b}$. Using the determinant formula, we have:

$$\boldsymbol{a} \wedge \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{vmatrix} = \hat{\boldsymbol{i}} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \hat{\boldsymbol{j}} \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} + \hat{\boldsymbol{k}} \begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix},$$

$$= \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}.$$

This vector is, of course, perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$, but it is not a unit vector. Its magnitude is $|\boldsymbol{a} \wedge \boldsymbol{b}| = \sqrt{1 + 4 + 9} = \sqrt{14}$. So the perpendicular unit vector is

$$\hat{\boldsymbol{n}} = \frac{1}{\sqrt{14}} \left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}} \right).$$

There is a second perpendicular unit vector pointing in the other direction $-\hat{\boldsymbol{n}}$, which is also an acceptable answer.

Now for the angle. Let's try to use the other formula for the vector product:

$$\boldsymbol{a} \wedge \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta\, \hat{\boldsymbol{n}}.$$

We want to find θ. The magnitudes are easy to compute: $|\boldsymbol{a}| = \sqrt{2^2 + 1} = \sqrt{5}$, and $|\boldsymbol{b}| = \sqrt{3^2 + 1} = \sqrt{10}$. Then we take the magnitude of both sides of the formula:

$$\begin{aligned} |\boldsymbol{a} \wedge \boldsymbol{b}| &= |\boldsymbol{a}||\boldsymbol{b}| \sin\theta\, |\hat{\boldsymbol{n}}|, \\ \sqrt{14} &= \sqrt{5} \cdot \sqrt{10} \cdot \sin\theta \cdot 1, \end{aligned}$$

noting that $\sin\theta$ is positive because θ is between 0 and π. And we solve for θ:

$$\theta = \sin^{-1}\left(\frac{\sqrt{7}}{5}\right) \approx 32^\circ,\ 148^\circ.$$

Note that the inverse sin is multivalued and yields two angles between 0 and π! This is a problem when trying to determine the angle between two vectors using the vector product. It is much safer to use the scalar product to get this angle, and also it is usually less work as well! Let's now do that. We have

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= |\boldsymbol{a}||\boldsymbol{b}| \cos\theta, \\ (2,1,0) \cdot (3,0,1) &= \sqrt{5}\sqrt{10}\cos\theta, \\ \rightarrow \cos\theta &= 6/\sqrt{50}, \end{aligned}$$

and thus $\theta = \cos^{-1}(6/\sqrt{50}) \approx 32^\circ$, on the range between 0 and π. So the smaller of the two angles is the correct one.

Example 1.10

A rigid body spins with angular speed ω about an axis $\hat{\boldsymbol{n}}$. Find a vector expression for the velocity of any point P in the body.

(The speed of the motion of any point is equal to the angular speed ω multiplied by the distance ρ between the point and the axis about which it is rotating.)

Let the velocity of point P be $\boldsymbol{u}$. Let us determine its magnitude first $|\boldsymbol{u}|$.

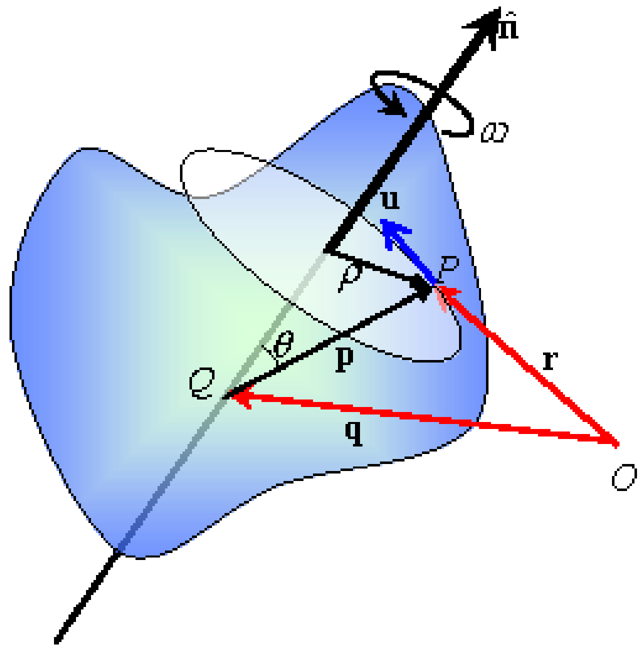


Figure 20: An irregularly shaped body spinning about the axis $\hat{\boldsymbol{n}}$ at a rate ω. Here P is a random point within the body and O is the origin.

From trigonometry we have

$$|\boldsymbol{u}| = \rho\omega = (|\boldsymbol{p}| \sin\theta)\omega = |\hat{\boldsymbol{n}} \wedge \boldsymbol{p}|\omega.$$

What about the direction of the motion, $\hat{\boldsymbol{u}}$? This direction is perpendicular to both the vector $\boldsymbol{p}$ and the rotation axis $\hat{\boldsymbol{n}}$, thus is parallel to $\hat{\boldsymbol{n}} \wedge \boldsymbol{p}$. The unit vector in this direction is hence $\hat{\boldsymbol{u}} = (\hat{\boldsymbol{n}} \wedge \boldsymbol{p})/|\hat{\boldsymbol{n}} \wedge \boldsymbol{p}|$. Putting everything together

$$\begin{aligned} \boldsymbol{u} = |\boldsymbol{u}|\hat{\boldsymbol{u}} &= (\hat{\boldsymbol{n}} \wedge \boldsymbol{p})\omega, \\ &= [\hat{\boldsymbol{n}} \wedge (\boldsymbol{r} - \boldsymbol{q})]\omega. \end{aligned}$$

1.7 Calculating distances with vectors

1.7.1 Shortest distance of a point from a line

Consider the line given by $\boldsymbol{r} = \boldsymbol{a} + \lambda(\boldsymbol{b} - \boldsymbol{a})$, and the point Q with position vector $\boldsymbol{q}$.

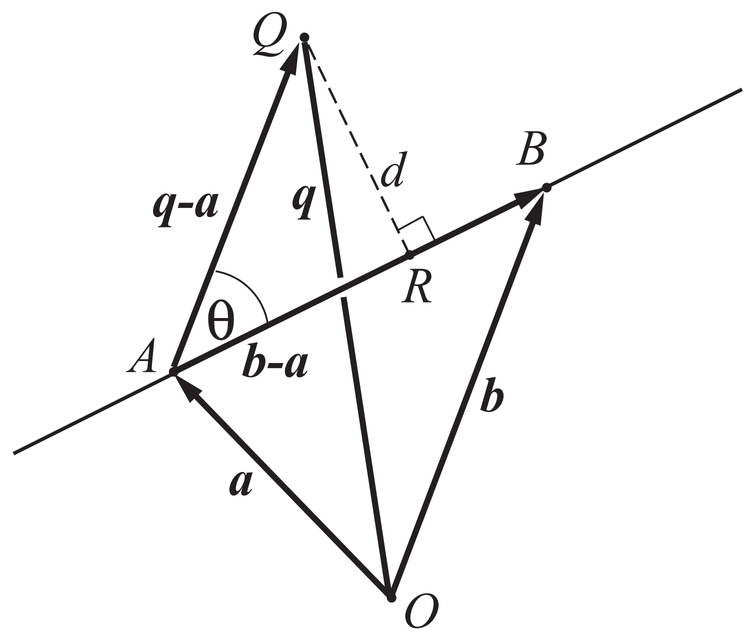


Figure 21: Shortest distance of the point Q from the line AB.

The shortest distance, d, between Q and the line is perpendicular to the line, i.e QR in the figure. Considering the right-angled triangle AQR, we see that

$$d = |\boldsymbol{q} - \boldsymbol{a}| \sin\theta \ ,$$

with θ the angle between $\overrightarrow{AQ} = \boldsymbol{q} - \boldsymbol{a}$ and $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$. Using the definition (37) of the cross product we can therefore write

$$\boxed{d = \frac{|(\boldsymbol{q} - \boldsymbol{a}) \wedge (\boldsymbol{b} - \boldsymbol{a})|}{|\boldsymbol{b} - \boldsymbol{a}|}} \, . \tag{52}$$

If we use as our equation of the line $\boldsymbol{r} = \boldsymbol{a} + \lambda\hat{\boldsymbol{t}}$, where $\hat{\boldsymbol{t}}$ is a unit vector, then (52) simplifies to

$$\boxed{d = |(\boldsymbol{q} - \boldsymbol{a}) \wedge \hat{\boldsymbol{t}}\,|} \, . \tag{53}$$

1.7.2 Shortest distance of a point from a plane

Consider the plane with equation $(\boldsymbol{r} - \boldsymbol{a}) \cdot \hat{\boldsymbol{n}} = 0$, and consider the point P with position vector $\boldsymbol{p}$.

- We draw the line (not necessarily in the plane) from P that is perpendicular

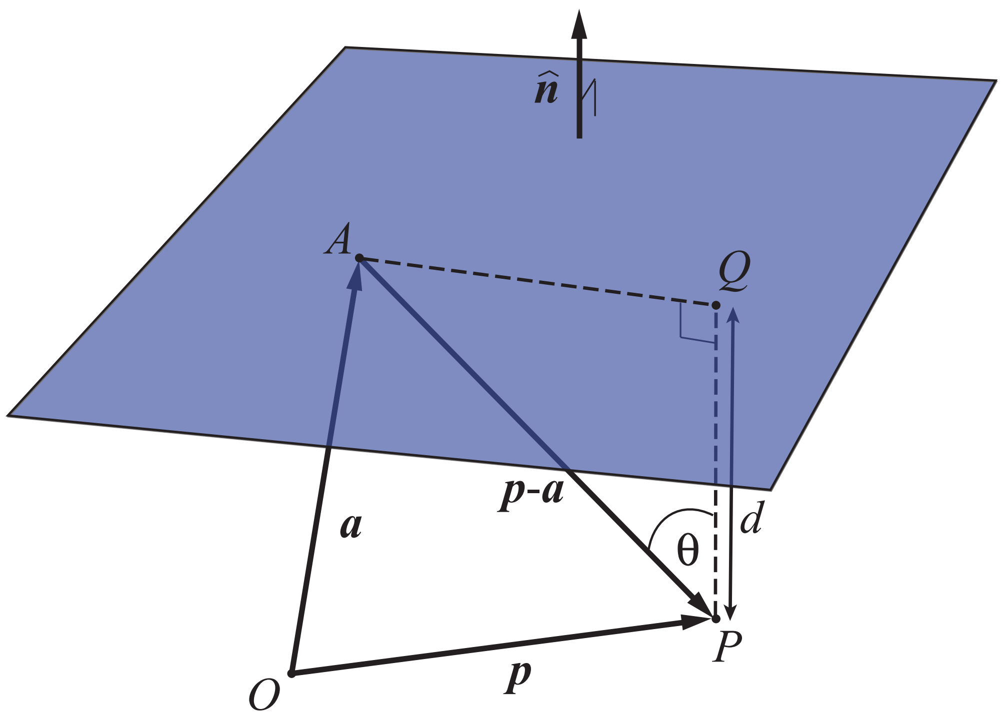


Figure 22: Shortest distance of the point P from a plane.

to the plane. This line meets the plane at Q. The shortest distance from the point P to the plane is the distance PQ.

- We consider the right-angled triangle APQ. Then

$$d \equiv |\overrightarrow{PQ}| = |\overrightarrow{AP}| \cos\theta \,, \tag{54}$$

and recalling the definition of the scalar product (23) we find that the shortest distance is given by

$$\boxed{d = |(\boldsymbol{p} - \boldsymbol{a}) \cdot \hat{\boldsymbol{n}}|} \,. \tag{55}$$

1.7.3 Shortest distance of a line from a line

Finally, we consider the shortest distance between two (non-parallel) lines L_1 and L_2, given by

$$\boldsymbol{r} = \boldsymbol{a}_1 + \lambda \hat{\boldsymbol{t}}_1, \qquad \text{and} \qquad \boldsymbol{r} = \boldsymbol{a}_2 + \mu \hat{\boldsymbol{t}}_2,$$

respectively, for λ and μ scalars.

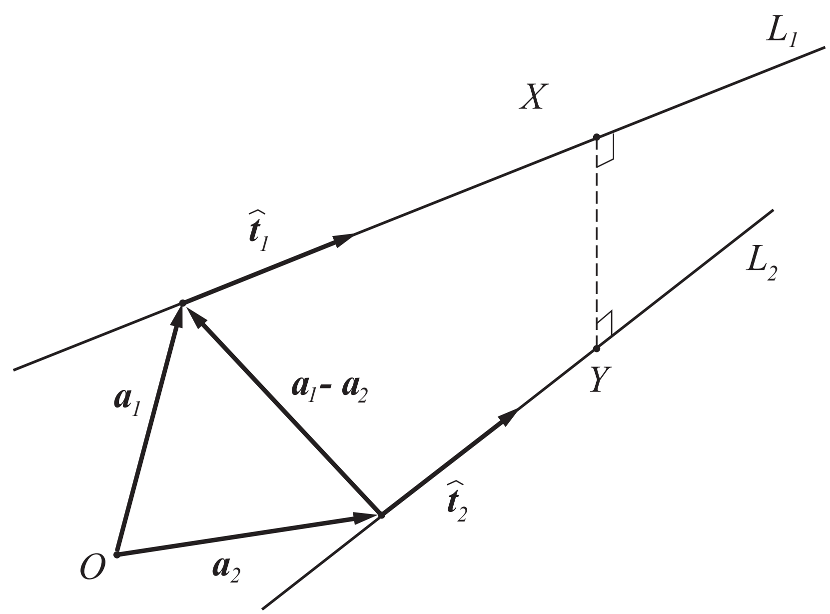


Figure 23: Non-intersecting lines L_1 and L_2 in three dimensions.

- Suppose that the shortest distance between the two lines is between the points X and Y. Since this is the shortest distance, the line joining X and Y will be perpendicular to both lines.

- The first question is: What direction is $\overrightarrow{XY}$ in? The two lines are parallel to $\hat{\boldsymbol{t}_1}$ and $\hat{\boldsymbol{t}_2}$ respectively, and $\overrightarrow{XY}$ is perpendicular to both of these. The vector which is perpendicular to both lines is simply $\boldsymbol{u} = \hat{\boldsymbol{t}_1} \wedge \hat{\boldsymbol{t}_2}$. We may then write $\overrightarrow{XY} = d\hat{\boldsymbol{u}}$, in which we have normalised $\boldsymbol{u}$ to get a unit vector. Our goal is to find d.

- Second thing: let us rewrite $\overrightarrow{XY}$ using the triangle XYO:

$$\overrightarrow{XY} = \overrightarrow{OY} - \overrightarrow{OX} = (\boldsymbol{a}_2 - \boldsymbol{a}_1) + \mu\hat{\boldsymbol{t}_2} - \lambda\hat{\boldsymbol{t}_1},$$

  for some specific scalars λ and μ (which we don't know yet).

- We have two expressions now for $\overrightarrow{XY}$. We equate them, then $\cdot\hat{\boldsymbol{u}}$ both sides of the equation. We obtain simply

$$\boxed{d = \frac{|(\boldsymbol{a}_2 - \boldsymbol{a}_1) \cdot (\hat{\boldsymbol{t}_1} \wedge \hat{\boldsymbol{t}_2})|}{|\hat{\boldsymbol{t}_1} \wedge \hat{\boldsymbol{t}_2}|}} \ . \tag{56}$$

Equation (56) gives us an easy way of finding out if two lines intersect or not. If they do intersect then their closest distance is obviously zero, i.e. $d = 0$. The condition for two lines to intersect is therefore

$$(\boldsymbol{a}_2 - \boldsymbol{a}_1) \cdot \hat{\boldsymbol{t}_1} \wedge \hat{\boldsymbol{t}_2} = 0 \ . \tag{57}$$

1.8 Scalar triple product

The quantity

$$\boldsymbol{a} \cdot (\boldsymbol{b} \wedge \boldsymbol{c}) \tag{58}$$

is a scalar, and is called the *scalar triple product* of $\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$. The *only* way that this product makes sense is if

- the vector product is taken first, to give the vector $\boldsymbol{b} \wedge \boldsymbol{c}$,
- and then the scalar product is taken between $\boldsymbol{a}$ and $\boldsymbol{b} \wedge \boldsymbol{c}$.

For this reason the brackets are unnecessary, and often omitted, so we write $\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c}$. Note also that the notation $\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c} = [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$ is sometimes used.

1.8.1 Component formula

In terms of components (instead of angles and magnitudes), the scalar triple product equals

$$\boxed{\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c} = a_x(b_y c_z - b_z c_y) + a_y(b_z c_x - b_x c_z) + a_z(b_x c_y - b_y c_x)}\,, \qquad (59)$$

where $\boldsymbol{a} = (a_x, a_y, a_z)$, etc.

The formula (59) is hard to remember, but can be expressed much more compactly using the notation of determinants introduced already. We write

$$\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c} \equiv \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \ , \tag{60}$$

which has been formed from taking the components of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ as the three rows. The determinant is now expanded using the pattern shown below (see also eq. (49)):

$$\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c} \ = \ a_x \begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} \ - \ a_y \begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix} \ + \ a_z \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} . \tag{61}$$

By rearranging this formula, we can find the following results:

$$\boxed{\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c} &= \ \boldsymbol{b} \cdot \boldsymbol{c} \wedge \boldsymbol{a} \ = \boldsymbol{c} \cdot \boldsymbol{a} \wedge \boldsymbol{b} \\ &= -\boldsymbol{a} \cdot \boldsymbol{c} \wedge \boldsymbol{b} = -\boldsymbol{b} \cdot \boldsymbol{a} \wedge \boldsymbol{c} = \ -\boldsymbol{c} \cdot \boldsymbol{b} \wedge \boldsymbol{a} \end{aligned}} \ . \tag{62}$$

In other words the value of $\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c}$ is unchanged if we make an $even$ permutation of the order of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, but the sign changes if we make an odd permutation of the order of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.

Permutations (aside)

- Suppose we have an ordered sequence of elements (such as $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$).
- A permutation is a rearrangement of the sequence in which the positions of some of the elements are swapped.
- An *even* permutation involves an *even* number of swaps of two elements, e.g. $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \;\rightarrow\; \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a}$ is even because we swap $\boldsymbol{a}$ and $\boldsymbol{c}$, and then swap $\boldsymbol{b}$ and $\boldsymbol{c}$ (two swaps). An *odd* permutation is when we make an *odd* number of swaps.
- For a sequence of three elements, an even permutation is equivalent to a *cyclic* permutation, e.g. $\boldsymbol{a} \rightarrow \boldsymbol{b}$, $\boldsymbol{b} \rightarrow \boldsymbol{c}$, $\boldsymbol{c} \rightarrow \boldsymbol{a}$.

Example 1.11 Evaluate the scalar triple product of the vectors $(2, 3, 1)$, $(1, 2, 2)$ and $(4, 5, 1)$.

We write the triple product in the form of a determinant:

$$\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \wedge \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \\ 4 & 5 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 2 & 2 \\ 5 & 1 \end{vmatrix} - 3 \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix},$$

$$= 2(2 \cdot 1 - 2 \cdot 5) + 3(2 \cdot 4 - 1 \cdot 1) + (1 \cdot 5 - 2 \cdot 4),$$

$$= 2.$$

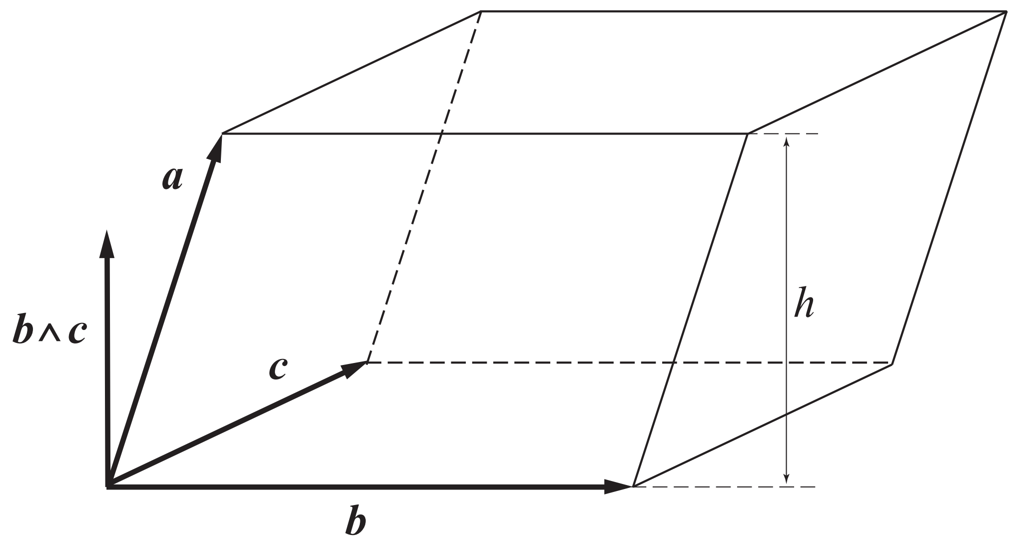


Figure 24: Parallelepiped formed from the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.

1.8.2 Parallelepipeds

One important application of the scalar triple product is the calculation of the volume of a *parallelepiped*.

Consider the parallelepiped formed by the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ (see Figure 24). The area of the base is

$$|\boldsymbol{b}||\boldsymbol{c}| \sin\theta = |\boldsymbol{b} \wedge \boldsymbol{c}| \ . \tag{63}$$

The height h of the parallelepiped corresponds to the component of the vector $\boldsymbol{a}$ in the direction perpendicular to the base.

The unit vector perpendicular to the base is

$$\frac{\boldsymbol{b} \wedge \boldsymbol{c}}{|\boldsymbol{b} \wedge \boldsymbol{c}|} \ , \tag{64}$$

and therefore the height h is

$$\boldsymbol{a} \cdot \frac{\boldsymbol{b} \wedge \boldsymbol{c}}{|\boldsymbol{b} \wedge \boldsymbol{c}|} \ . \tag{65}$$

The volume of the parallelepiped is the area of its base times its height, and putting together (63) and (65) we find that the volume is

$$\text{Volume} = |\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c}| \ , \tag{66}$$

i.e. the scalar triple product.

Example 1.12 Show that $\boldsymbol{a} \cdot \boldsymbol{a} \wedge \boldsymbol{c} = \boldsymbol{0}$. What does this result mean geometrically? What is the condition for three vectors to be coplanar?

We just permute the elements in the triple product:

$$\boldsymbol{a} \cdot \boldsymbol{a} \wedge \boldsymbol{c} = -\boldsymbol{c} \cdot \boldsymbol{a} \wedge \boldsymbol{a} = 0$$

the last equality coming from the fact that a vector product of a vector with itself is zero. Of course, this is all obvious because a vector product involving $\boldsymbol{a}$ will always give you a vector perpendicular to $\boldsymbol{a}$.

A geometric interpretation of this is that the parallelepiped formed from $\boldsymbol{a}$, $\boldsymbol{a}$ and $\boldsymbol{c}$ has zero volume because it is necessarily flat, i.e. a 2D object.

We can then use this idea to derive a test to see if any three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are coplanar. If the three vectors are coplanar (i.e. lying in the same plane) then the parallelepiped they form is flat and has zero volume, or in mathematical terms:

$$\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c} = \boldsymbol{0}.$$

Example 1.13 Find the volume of the rhomboid drawn in the face-centred cubic in Figure 25.

A rhomboid is a special case of a parallelepiped. Let us work out the three vectors that define it. First note that $\overrightarrow{OA} = (a, 0, 0)$, $\overrightarrow{OB} = (a/2, a/2, 0)$, $\overrightarrow{OC} = (a, a/2, a/2)$, and $O\vec{D} = (a/2, 0, a/2)$. The three defining vectors of

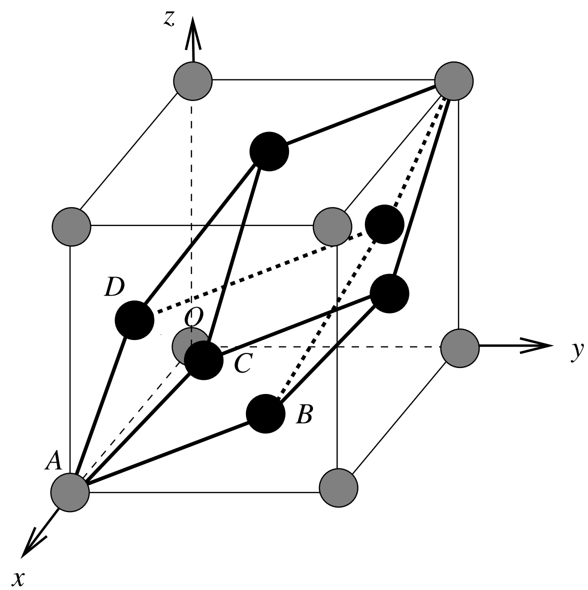


Figure 25: Cubic lattice, side length a. The black atoms are located at the centres of each face and the grey atoms at the corners.

the rhomboid are:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (-a/2, a/2, 0),$$
$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (0, a/2, a/2),$$
$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (-a/2, 0, a/2).$$

The volume is just the scalar triple product of these vectors:

$$\overrightarrow{AB} \cdot \overrightarrow{AC} \wedge \overrightarrow{AD} = \begin{vmatrix} -a/2 & a/2 & 0 \\ 0 & a/2 & a/2 \\ -a/2 & 0 & a/2 \end{vmatrix},$$

$$= -(a/2) \begin{vmatrix} a/2 & a/2 \\ 0 & a/2 \end{vmatrix} - (a/2) \begin{vmatrix} 0 & a/2 \\ -a/2 & a/2 \end{vmatrix} + 0,$$

$$= -(a/2)(a/2)^2 - (a/2)(a/2)^2 = -a^3/4.$$

The volume is the absolute value of this (the minus sign comes about because the three vectors defining the rhomboid are left handed).

1.9 Vector triple product

The quantity $\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c})$ is a vector, and is called the *vector triple product* of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. Unlike $\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c}$, the position of the brackets in $\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c})$ is crucial!

We can use our expression for the vector product in components to expand $\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c})$ in components. First, use (46) to evaluate $\boldsymbol{b} \wedge \boldsymbol{c}$, so that

$$\begin{aligned}\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c}) = (a_x\hat{\boldsymbol{i}} + a_y\hat{\boldsymbol{j}} + a_z\hat{\boldsymbol{k}}) \wedge \Big[(b_y c_z - b_z c_y)\hat{\boldsymbol{i}} \\ +(b_z c_x - b_x c_z)\hat{\boldsymbol{j}} + (b_x c_y - b_y c_x)\hat{\boldsymbol{k}}\Big]\end{aligned}$$

and then using the distributive property (39) to expand the brackets and the results (41) and (42) to evaluate the cross products, we find

$$\begin{aligned}\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c}) &= (-a_y\hat{\boldsymbol{k}} + a_z\hat{\boldsymbol{j}})(b_y c_z - b_z c_y) + (a_x\hat{\boldsymbol{k}} - a_z\hat{\boldsymbol{i}})(b_z c_x - b_x c_z) \\ &+ (-a_x\hat{\boldsymbol{j}} + a_y\hat{\boldsymbol{i}})(b_x c_y - b_y c_x) \ .\end{aligned}$$

gathering the terms in $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ we obtain the final result

$$\boxed{\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}} \ . \tag{67}$$

Essentially, this is a kind of projection of $\boldsymbol{a}$ down onto the plane formed by the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$.

We can proceed in exactly the same manner to find $(\boldsymbol{a} \wedge \boldsymbol{b}) \wedge \boldsymbol{c}$ in the form

$$\boxed{(\boldsymbol{a} \wedge \boldsymbol{b}) \wedge \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}} \, . \tag{68}$$

By comparing (67) and (68) it becomes clear that $\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c}) \neq (\boldsymbol{a} \wedge \boldsymbol{b}) \wedge \boldsymbol{c}$ in general.

<u>Example 1.14</u> Calculate $(\hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{j}}) \wedge \hat{\boldsymbol{i}}$ in two ways. Calculate $(\boldsymbol{a} \wedge \boldsymbol{b}) \wedge (\boldsymbol{c} \wedge \boldsymbol{d})$.

One way uses the identities $\hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{j}} = \hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{i}} = \hat{\boldsymbol{j}}$. Simply:

$$(\hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{j}}) \wedge \hat{\boldsymbol{i}} = \hat{\boldsymbol{j}}.$$

A different way uses the general identity for triple vector products:

$$(\hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{j}}) \wedge \hat{\boldsymbol{i}} = (\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{i}})\hat{\boldsymbol{j}} - (\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{j}})\hat{\boldsymbol{i}} = \hat{\boldsymbol{j}}$$

A third way computes the vector products using the determinant formula and coordinates (noting that $\hat{\boldsymbol{j}} = (0, 1, 0)$, etc.) This is a boring way, which we will leave to one side.

The last part of the question deals with a daunting vector quadruple product. This can be transformed into a triple product by setting $\boldsymbol{r} = \boldsymbol{c} \wedge \boldsymbol{d}$, then

$$\begin{aligned}(\boldsymbol{a} \wedge \boldsymbol{b}) \wedge (\boldsymbol{c} \wedge \boldsymbol{d}) &= (\boldsymbol{a} \wedge \boldsymbol{b}) \wedge \boldsymbol{r}, \\ &= (\boldsymbol{a} \cdot \boldsymbol{r})\boldsymbol{b} - (\boldsymbol{r} \cdot \boldsymbol{b})\boldsymbol{a}, \\ &= [\boldsymbol{a} \cdot (\boldsymbol{c} \wedge \boldsymbol{d})]\,\boldsymbol{b} - [\boldsymbol{b} \cdot (\boldsymbol{c} \wedge \boldsymbol{d})]\,\boldsymbol{a},\end{aligned}$$

an expression that involves two scalar triple products.

But equally we could have set $\boldsymbol{s} = \boldsymbol{a} \wedge \boldsymbol{b}$ and proceeded to derive a similar but different expression:

$$\begin{aligned}(\boldsymbol{a} \wedge \boldsymbol{b}) \wedge (\boldsymbol{c} \wedge \boldsymbol{d}) &= \boldsymbol{s} \wedge (\boldsymbol{c} \wedge \boldsymbol{d}), \\ &= (\boldsymbol{s} \cdot \boldsymbol{d})\boldsymbol{c} - (\boldsymbol{s} \cdot \boldsymbol{c})\boldsymbol{d}, \\ &= [\boldsymbol{d} \cdot (\boldsymbol{a} \wedge \boldsymbol{b})]\,\boldsymbol{c} - [\boldsymbol{c} \cdot (\boldsymbol{a} \wedge \boldsymbol{b})]\,\boldsymbol{d}.\end{aligned}$$

We can equate these two expressions and get something like:

$$\alpha \boldsymbol{a} + \beta \boldsymbol{b} = \gamma \boldsymbol{c} + \delta \boldsymbol{d}$$

where α, β, γ and δ are scalar triple products. This is basically saying that, in 3D, there must be a linear relation between any four vectors. In other words, given four vectors in 3D, you can generally write one of those vectors in terms of the other three.

1.10 Vector area

We are familiar with the concept of the area of a surface as being a scalar. However, in some applications, particularly with integrals of vector quantities such as electric or magnetic flux through a surface, we will also want to think about the *vector area*.

Consider a flat planar surface of area A, with unit normal $\hat{\boldsymbol{n}}$, then the vector area $\boldsymbol{S}$ is defined by

$$\boxed{\boldsymbol{S} = A\hat{\boldsymbol{n}}} \; . \tag{69}$$

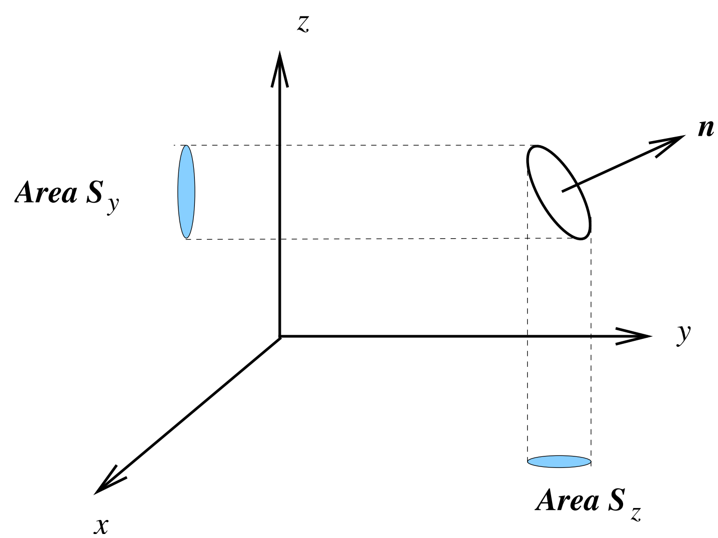


Figure 26: Vector area $\boldsymbol{S}$ and its projections on to the xz and xy planes.

In terms of Cartesian components, we can write

$$\begin{aligned} \boldsymbol{S} &\equiv S_x\hat{\boldsymbol{i}} + S_y\hat{\boldsymbol{j}} + S_z\hat{\boldsymbol{k}} \\ &= A\Big[(\hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{n}})\hat{\boldsymbol{i}} + (\hat{\boldsymbol{j}}\cdot\hat{\boldsymbol{n}})\hat{\boldsymbol{j}} + (\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{n}})\hat{\boldsymbol{k}}\Big] \end{aligned} \tag{70}$$

where the second step comes from simply taking the scalar product of $\boldsymbol{S}$ with each axis in turn. The components S_x, S_y, S_z are also just the areas of the projections of the patch onto the planes $x = 0, y = 0$, and $z = 0$, respectively.

Given any surface, there is an ambiguity in which direction the normal should point. However, for a closed surface, such as a cube, one always takes the enclosing surfaces to have normals pointing *out* of the volume enclosed.

The vector area of the whole surface is then taken to be the sum over the vector areas of each individual patch. **For a closed surface it turns out the vector area is zero.** (We will not prove this result here.)

We can also approximate the vector area of a *curved* surface (such as a sphere)

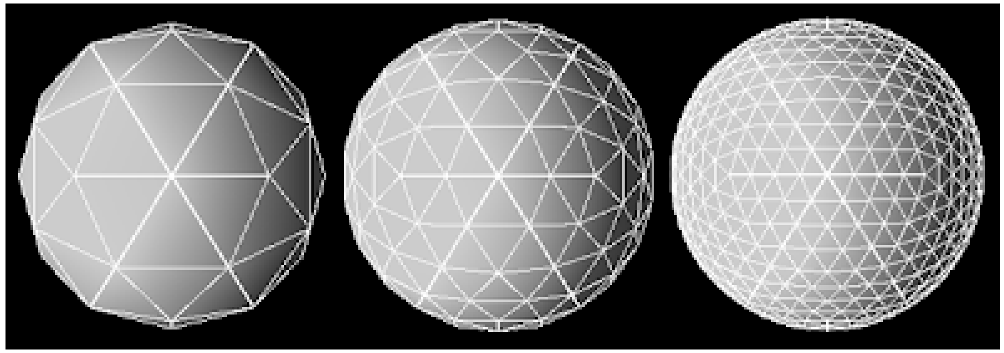
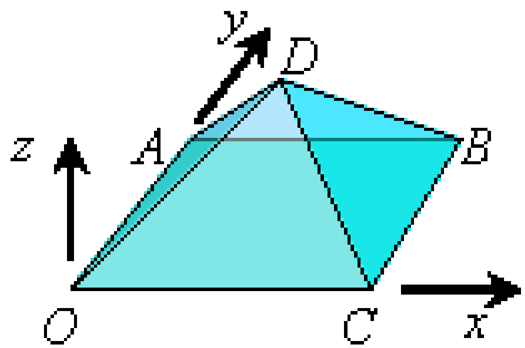


Figure 27: Successive polyhedral approximations to the surface of a sphere

by splitting the surface into small plane patches. In the limit of these patches shrinking to zero size the finite sum is then replaced by an integral over the whole surface to give the exact vector area. Calculating such surface integrals will be described in the course next term, but certain examples are possible now using the fact that the vector area of a closed surface vanishes.

Example 1.16



Calculate the vector area of the pyramid with vertices $O = (0,0,0)$, $A = (0,1,0)$, $B = (1,1,0)$, $C = (1,0,0)$ and D = $(\frac{1}{2},\frac{1}{2},1)$, excluding the base.

Let us first work out the vector area of the triangle OAD. To get the normal to this surface, we calculate $\boldsymbol{n} = \overrightarrow{OD} \wedge \overrightarrow{OA}$:

$$\boldsymbol{n} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 1/2 & 1/2 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\hat{\boldsymbol{i}} + \frac{1}{2}\hat{\boldsymbol{k}}. \tag{71}$$

But we want the unit normal. The magnitude of $\boldsymbol{n}$ is $\sqrt{1+1/4} = \sqrt{5}/2$, and so the unit normal to this surface is:

$$\hat{\boldsymbol{n}} = \frac{2}{\sqrt{5}}(-1, 0, \tfrac{1}{2}).$$

What is the scalar area of OAD? This is actually

$$(1/2)|\overrightarrow{OD} \wedge \overrightarrow{OA}| = (1/2)|\boldsymbol{n}| = \frac{\sqrt{5}}{4}.$$

So finally the vector area of OAD is

$$\boldsymbol{S}_{\mathsf{OAD}} = \frac{\sqrt{5}}{4}\hat{\boldsymbol{n}} = \left(-\frac{1}{2}, 0, \frac{1}{4}\right).$$

Doing similar calculations we can find the vector area of the other triangular faces: OCD, CBD, and ABD:

$$\boldsymbol{S}_{\mathsf{OCD}} = (0, -\tfrac{1}{2}, \tfrac{1}{4}), \quad \boldsymbol{S}_{\mathsf{CBD}} = (\tfrac{1}{2}, 0, \tfrac{1}{4}), \quad \boldsymbol{S}_{\mathsf{ABD}} = (0, \tfrac{1}{2}, \tfrac{1}{4}).$$

The total vector area of the pyramid (without base) is the sum of these four triangular vector areas. We find that $\mathbf{S} = (0,0,1) = \hat{\boldsymbol{k}}$.

Alternatively: if we do include the base, then the total vector area of the closed pyramidal surface is zero, i.e.

$$\boldsymbol{S}_{\text{pyramid}} + \boldsymbol{S}_{\text{base}} = \mathbf{0}.$$

The vector area of the base is easy: its scalar area is 1 and it points vertically downwards. Thus $\boldsymbol{S}_{\text{base}} = -\hat{\boldsymbol{k}}$, and so $\boldsymbol{S}_{\text{pyramid}} = \hat{\boldsymbol{k}}$.

Example 1.17 Calculate the vector area of a hemispherical shell of radius a, oriented so that its circular base lies in the xy plane and the shell itself is above the plane.

The total vector area of the hemisphere and its circular base is *zero*, because such a figure is a closed volume:

$$\boldsymbol{S}_{\text{total}} = \boldsymbol{S}_{\text{hemi}} + \boldsymbol{S}_{\text{base}} = \mathbf{0}.$$

The vector area of the hemisphere is difficult to calculate directly, but the base is easy as it is a circle of radius a and its normal pointing out of the volume is $-\hat{\boldsymbol{k}}$. Therefore: $\boldsymbol{S}_{\text{base}} = -\pi a^2 \hat{\boldsymbol{k}}$, which means that $\boldsymbol{S}_{\text{hemi}} = \pi a^2 \hat{\boldsymbol{k}}$.

1.11 General basis vectors

In three-dimensional space any three *non-coplanar* vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ constitute a *basis*

This means that any vector can be expressed as a linear combination of these three basis vectors

$$\boldsymbol{u} = \alpha \boldsymbol{a} + \beta \boldsymbol{b} + \gamma \boldsymbol{c},$$

where $\boldsymbol{u}$ is any vector and α, β, and γ are unique scalars: they are called $\boldsymbol{u}$'s components with respect to this specific basis.

It is not necessary for the basis vectors to be perpendicular to each other. However, if they are, the basis is said to be **orthogonal**, and if the basis vectors have unit length as well then the basis is **orthonormal**. The Cartesian basis $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$, $\hat{\boldsymbol{k}}$ is orthonormal.

In the special situation of an orthogonal basis, α, β, and γ can be obtained by the scalar product. We find $\alpha = (\boldsymbol{u} \cdot \boldsymbol{a})/|\boldsymbol{a}|^2$, $\beta = (\boldsymbol{u} \cdot \boldsymbol{b})/|\boldsymbol{b}|^2$, $\gamma = (\boldsymbol{u} \cdot \boldsymbol{c})/|\boldsymbol{c}|^2$. However, for non-orthogonal basis sets this is not true and the components α, β, and γ can be obtained in terms of scalar triple products (see worked example later).

1.12 Other orthogonal coordinate systems

The Cartesian xyz coordinates are a very familiar example of an orthogonal coordinate system in three dimensions. However, there are other examples which can be very useful in particular applications, and we will describe the two most commonly used ones here.

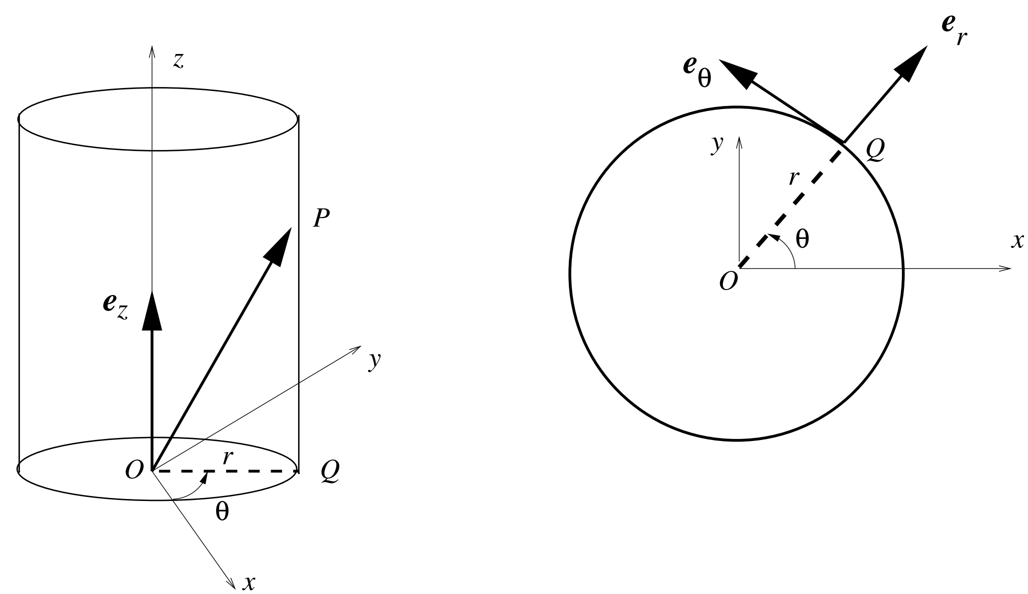


Figure 28: Diagrams illustrating the cylindrical polar coordinate system.

1.12.1 Cylindrical polar coordinates

We introduce the cylindrical polar coordinates (r, θ, z).

- The z coordinate is the same as for Cartesian coordinates,

- the r coordinate corresponds to the distance OQ
- the azimuthal angle θ corresponds to the angle between the x axis and the vector $\overrightarrow{OQ}$. By convention we take $0 \leqslant \theta < 2\pi$, but an alternative is $-\pi < \theta \leqslant \pi$.
- The general point P has position vector $\boldsymbol{r} = (x, y, z)$ in Cartesians. Via simple trigonometry its x and y components are

$$x = r\cos\theta \qquad\qquad y = r\sin\theta \;, \tag{72}$$

so that

$$\boxed{\boldsymbol{r} = r\cos\theta\hat{\boldsymbol{\imath}} + r\sin\theta\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}} \;. \tag{73}$$

- As well as new coordinates (r, θ, z) we introduce new coordinate directions, or rather unit vectors $\hat{\boldsymbol{e}}_{\boldsymbol{r}}$,$\hat{\boldsymbol{e}}_{\boldsymbol{\theta}}$, $\hat{\boldsymbol{e}}_{\boldsymbol{z}}$, defined via

$$\boxed{\begin{aligned} \hat{\boldsymbol{e}}_{\boldsymbol{r}} &= \cos\theta\hat{\boldsymbol{i}} + \sin\theta\hat{\boldsymbol{j}} \\ \hat{\boldsymbol{e}}_{\boldsymbol{\theta}} &= -\sin\theta\hat{\boldsymbol{i}} + \cos\theta\hat{\boldsymbol{j}} \\ \hat{\boldsymbol{e}}_{\boldsymbol{z}} &= \hat{\boldsymbol{k}} \end{aligned}} \tag{74}$$

The three unit vectors $\hat{\boldsymbol{e}}_{\boldsymbol{r}}$,$\hat{\boldsymbol{e}}_{\boldsymbol{\theta}}$, $\hat{\boldsymbol{e}}_{\boldsymbol{z}}$ are perpendicular to each other (check that the scalar products are zero) and form a right-handed orthonormal system. In particular (check using 74)

$$\hat{\boldsymbol{e}}_{\boldsymbol{z}} = \hat{\boldsymbol{e}}_{\boldsymbol{r}} \wedge \hat{\boldsymbol{e}}_{\boldsymbol{\theta}} \quad \hat{\boldsymbol{e}}_{\boldsymbol{r}} = \hat{\boldsymbol{e}}_{\boldsymbol{\theta}} \wedge \hat{\boldsymbol{e}}_{\boldsymbol{z}} \quad \hat{\boldsymbol{e}}_{\boldsymbol{\theta}} = \hat{\boldsymbol{e}}_{\boldsymbol{z}} \wedge \hat{\boldsymbol{e}}_{\boldsymbol{r}} \ .$$

Watch out, these vectors are not constant vectors! They depend on position!

Finally,

$$|\boldsymbol{r}| = \sqrt{r^2 + z^2} \ , \tag{75}$$

and in terms of the new cylindrical basis vectors (rather than the Cartesian basis vectors):

$$\boldsymbol{r} = r\hat{\boldsymbol{e}}_{\boldsymbol{r}} + z\hat{\boldsymbol{e}}_{\boldsymbol{z}} \ . \tag{76}$$

1.12.2 Plane polar coordinates

If we just work in the xy plane, i.e. in two dimensions, then we can use plane polar coordinates (r, θ). These are simply cylindrical polar coordinates but with the z direction suppressed. So for instance in two dimensions

$$\boxed{\boldsymbol{r} = r\cos\theta\hat{\boldsymbol{\imath}} + r\sin\theta\hat{\boldsymbol{\jmath}}} \ . \tag{77}$$

We have seen this 2D parametrisation already in eq. (11).

1.12.3 Spherical coordinates

We next introduce the spherical polar coordinates (r, θ, ϕ).

- The r coordinate is the distance of the point P from the origin.

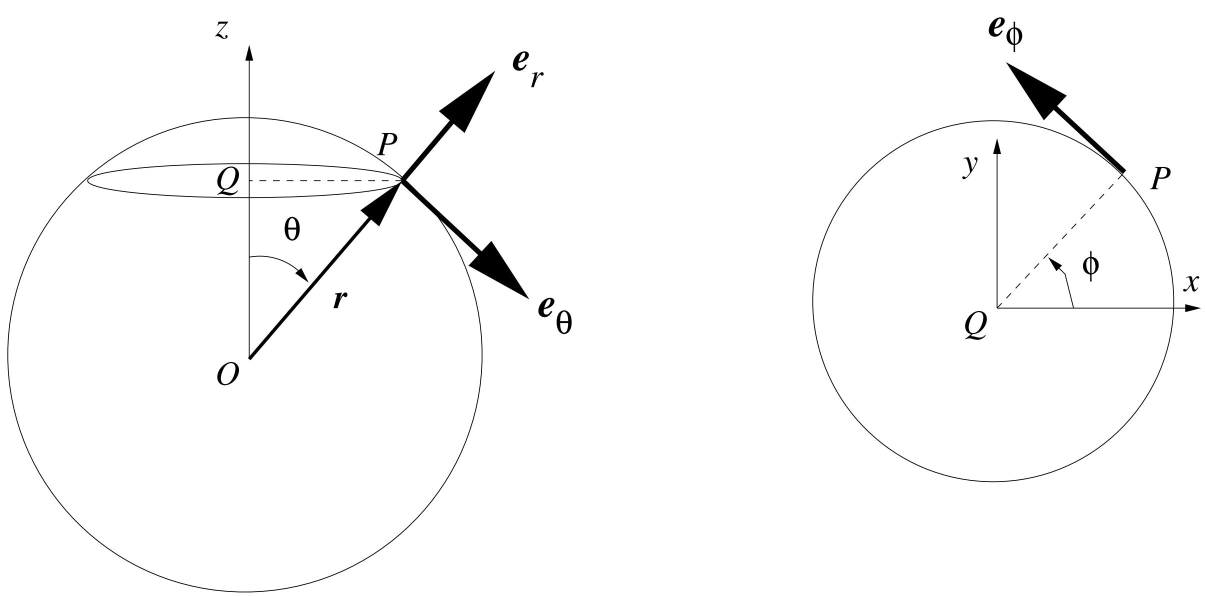


Figure 29: Diagrams illustrating the spherical coordinate system.

- The co-latitude θ coordinate is the angle between the upward vertical axis and the line $\overrightarrow{OP}$, measured in the direction *down* from the North Pole. So that the North Pole has $\theta = 0$, the equator has $\theta = \pi/2$, and the South Pole has $\theta = \pi$.

- The azimuthal angle ϕ coordinate is the angle measured from the fixed x axis round to the direction $\overrightarrow{QP}$ in the anticlockwise sense.

- It is important to note the ranges of the angular coordinates:

$$\boxed{\begin{array}{ccccc} 0 & \leqslant & \theta & \leqslant & \pi \\ 0 & \leqslant & \phi & < & 2\pi \end{array}}\,.$$

- Note that $\boldsymbol{r}$, $\boldsymbol{\theta}$ in spherical polars are different from $\boldsymbol{r}$, $\boldsymbol{\theta}$ in cylindrical polars (with the latter often denoted $\boldsymbol{\rho}$, $\boldsymbol{\phi}$ to avoid confusion).

- The general point P has position vector $\boldsymbol{r}$; from the right-angled triangle

OPQ we see that

$$|PQ| = r\sin\theta \quad , \qquad |OQ| = z = r\cos\theta \; . \tag{78}$$

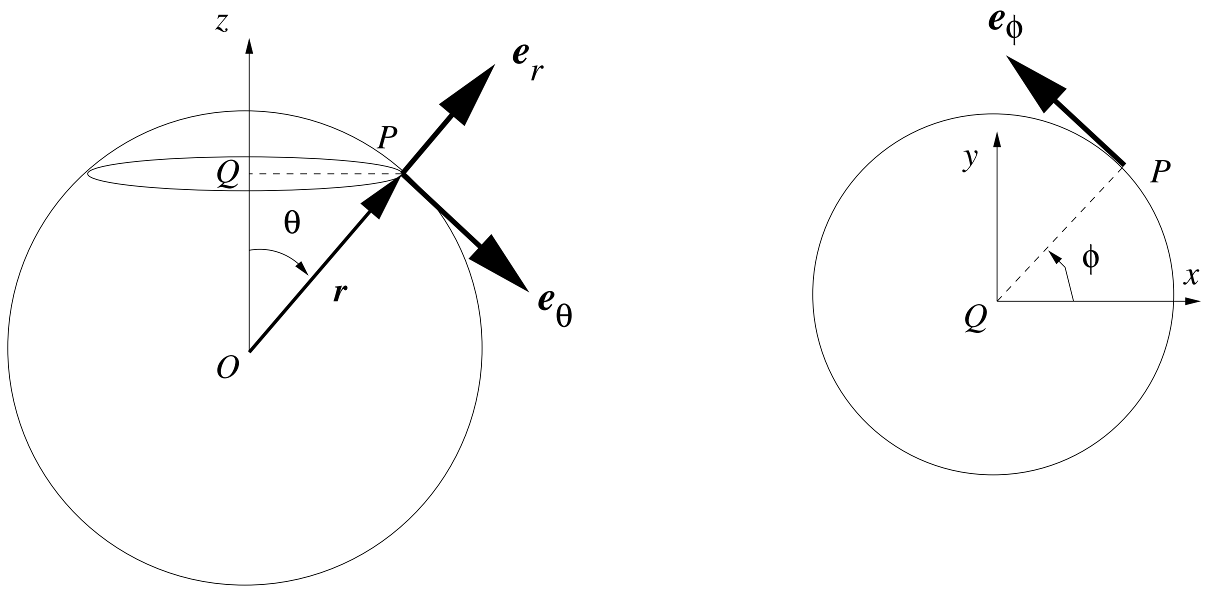


Figure 30: Diagrams illustrating the spherical coordinate system.

Then, via simple trigonometry in the xy plane

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \end{aligned}, \tag{79}$$

so that

$$\boxed{\boldsymbol{r} = r\sin\theta\cos\phi\hat{\boldsymbol{\imath}} + r\sin\theta\sin\phi\hat{\boldsymbol{\jmath}} + r\cos\theta\hat{\boldsymbol{k}}} \, . \tag{80}$$

- We can introduce associated unit vectors: $\hat{\boldsymbol{e}}_{\boldsymbol{r}}$,$\hat{\boldsymbol{e}}_{\boldsymbol{\theta}}$, $\hat{\boldsymbol{e}}_{\boldsymbol{\phi}}$. These point in the direction of increasing r, θ, and ϕ, respectively.

  The spherical polar unit vectors can be written in terms of $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$. To do this, consider the (vertical) plane made by the three points O, P, and Q, with O as origin. The vertical unit vector in this plane is $\hat{\boldsymbol{k}}$, and the horizontal unit vector is $\hat{\boldsymbol{u}} = \overrightarrow{QP}/|\overrightarrow{QP}| = \cos\phi\hat{\boldsymbol{\imath}} + \sin\phi\hat{\boldsymbol{\jmath}}$.

  The unit radial vector in the plane is now $\hat{\boldsymbol{e}}_{\boldsymbol{r}} = \cos(\frac{\pi}{2} - \theta)\hat{\boldsymbol{u}} + \sin(\frac{\pi}{2} - \theta)\hat{\boldsymbol{k}}$.

And similarly $\hat{\boldsymbol{e}}_{\boldsymbol{\theta}} = \sin(\frac{\pi}{2} - \theta)\hat{\boldsymbol{u}} - \cos(\frac{\pi}{2} - \theta)\hat{\boldsymbol{k}}$. These may be rewritten as

$$\boxed{\begin{aligned} \hat{\boldsymbol{e}}_{\boldsymbol{r}} &= \sin\theta\cos\phi\hat{\boldsymbol{\imath}} + \sin\theta\sin\phi\hat{\boldsymbol{\jmath}} + \cos\theta\hat{\boldsymbol{k}} \\ \hat{\boldsymbol{e}}_{\boldsymbol{\theta}} &= \cos\theta\cos\phi\hat{\boldsymbol{\imath}} + \cos\theta\sin\phi\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{k}} \end{aligned}} \tag{81}$$

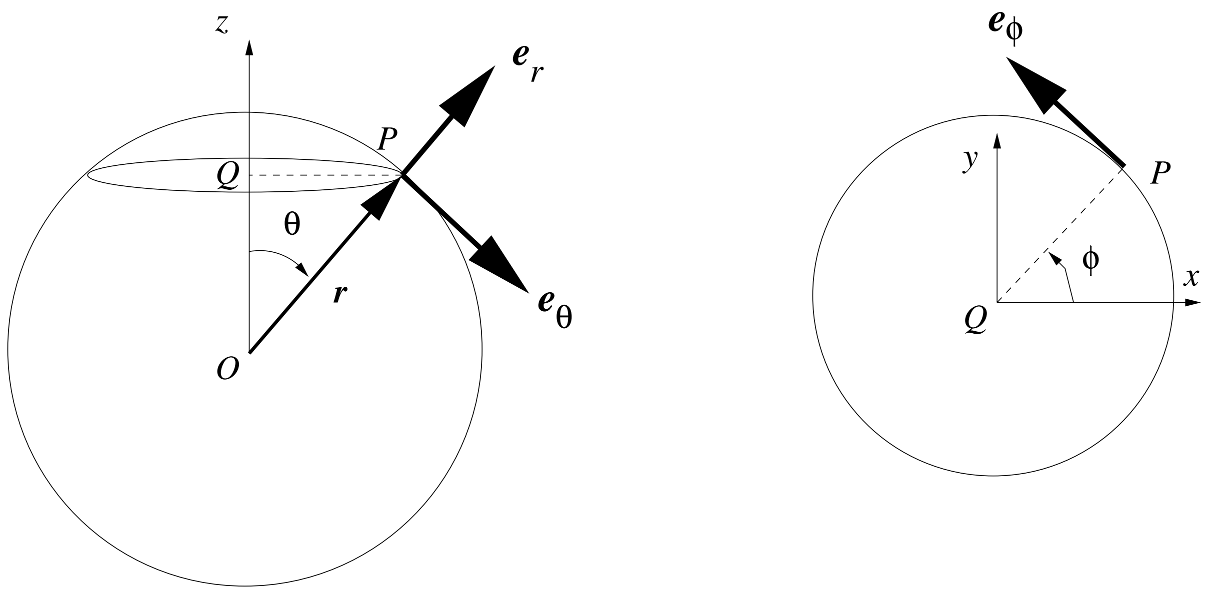


Figure 31: Diagrams illustrating the spherical coordinate system.

Finally, in the xy plane we resolve $\hat{\boldsymbol{e}}_\phi$ in the directions of $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ to find

$$\boxed{\hat{\boldsymbol{e}}_{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{\imath}} + \cos\phi\,\hat{\boldsymbol{\jmath}}} \tag{82}$$

- The three unit vectors $\hat{\boldsymbol{e}}_{\boldsymbol{r}}$,$\hat{\boldsymbol{e}}_{\boldsymbol{\theta}}$, $\hat{\boldsymbol{e}}_\phi$ are perpendicular to each other, and form a right-handed orthonormal system. In particular,

$$\hat{\boldsymbol{e}}_\phi = \hat{\boldsymbol{e}}_{\boldsymbol{r}} \wedge \hat{\boldsymbol{e}}_{\boldsymbol{\theta}} \quad \hat{\boldsymbol{e}}_{\boldsymbol{r}} = \hat{\boldsymbol{e}}_{\boldsymbol{\theta}} \wedge \hat{\boldsymbol{e}}_\phi \quad \hat{\boldsymbol{e}}_{\boldsymbol{\theta}} = \hat{\boldsymbol{e}}_\phi \wedge \hat{\boldsymbol{e}}_{\boldsymbol{r}} \ .$$

  Finally, note that in spherical polar basis vectors the position vector of P is very simple:

$$\boldsymbol{r} = r\hat{\boldsymbol{e}}_{\boldsymbol{r}} \ . \tag{83}$$

Example 1.18 Calculate the coordinates of the point $(1, 1, 2)$ in (a) cylindrical polars, (b) spherical polars.

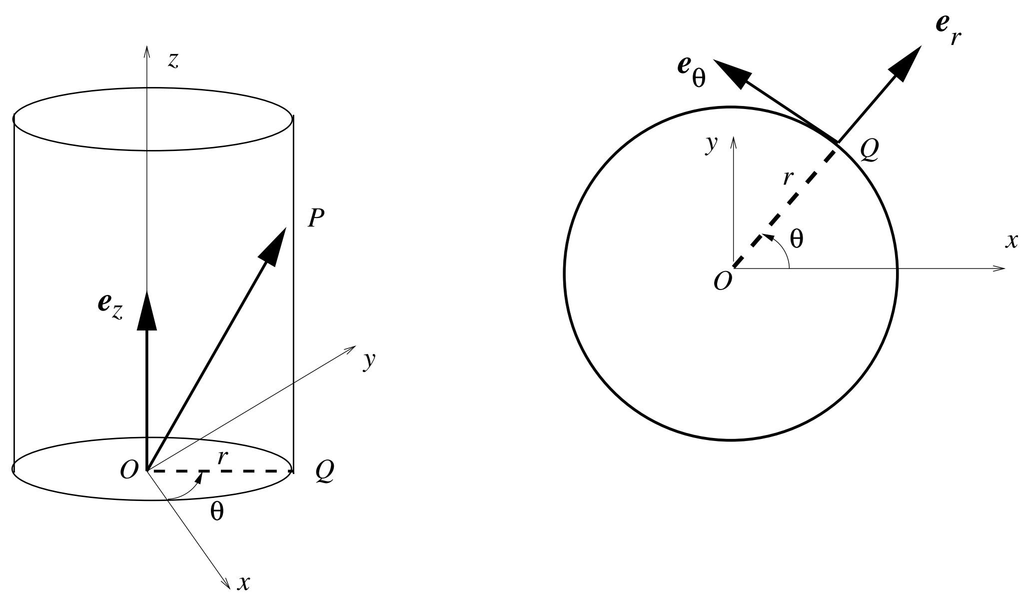


Figure 32: Diagrams illustrating the cylindrical polar coordinate system.

(a) In cylindrical polars, radius is given by $r^2 = x^2 + y^2 = 2$ and so $r = \sqrt{2}$, whereas azimuthal angle is given by $\tan\theta = y/x = 1$, which means $\theta = \pi/4$. The vertical height z is unchanged.

(b) In spherical polars, radius is given by $r^2 = x^2 + y^2 + z^2 = 1 + 1 + 2^2$ and so $r = \sqrt{6}$.

We also have $z = r\cos\theta$ which lets us get the colatitude θ: $\cos\theta = 2/\sqrt{6}$, which means $\theta = \cos^{-1}\sqrt{2/3} \approx 35°$.

Finally, $x = r\sin\theta\cos\phi$, which yields the azimuthal angle ϕ:

$$\cos\phi = \frac{x}{r\sin\theta} = \frac{x}{r\sqrt{1-\cos^2\theta}} = \frac{1}{\sqrt{6}\sqrt{1-2/3}} = \frac{1}{\sqrt{2}},$$

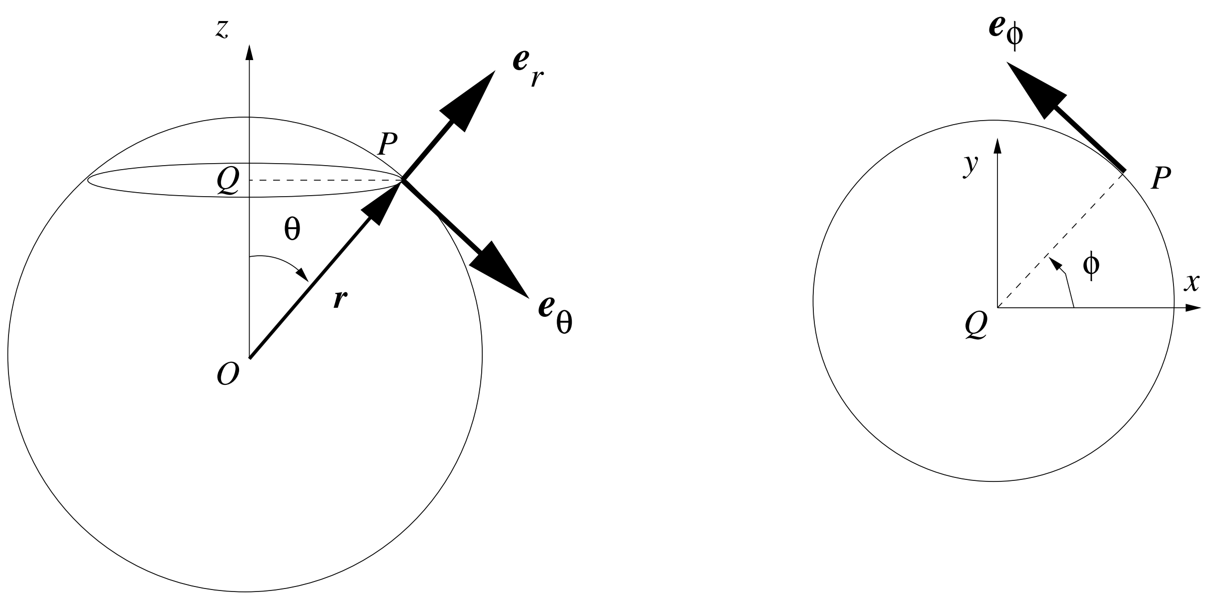


Figure 33: Diagrams illustrating the spherical coordinate system.

and so $\phi = \cos^{-1}\sqrt{1/2} = \pi/4 = 45^\circ$.

Example 1.19 [Tripos question 2005 Paper 1, 5C]

The vertices of a tetrahedron O, P, Q, R have coordinates $(0,0,0)$, $(2,1,1)$, $(1,2,2)$ and $(0,0,3)$ respectively. Find by vector methods:

(a) the angle between the faces OPR and OQR

(b) the angle between the vector OP and the normal to the face PQR

(c) the area of the face PQR

(d) the shortest distance from the origin to the plane containing P, Q and R.

(a) The triangular surfaces OPR and OQR define two planes with unit normals $\hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{n}}_2$, respectively. The angle between two planes is equal to the angle between their normals. So let us find $\hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{n}}_2$ and then get the angle between them.

First define

$$\boldsymbol{n}_1 = \overrightarrow{OP} \times \overrightarrow{OR} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 2 & 1 & 1 \\ 0 & 0 & 3 \end{vmatrix}$$

$$= \hat{\boldsymbol{i}} \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} - \hat{\boldsymbol{j}} \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} + \hat{\boldsymbol{k}} \begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix},$$

$$= 3\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}}.$$

Its magnitude is $\sqrt{3^2 + (-6)^2} = 3\sqrt{5}$, and so $\hat{\boldsymbol{n}}_1 = (\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}})/\sqrt{5}$.

Similarly, we find $\boldsymbol{n}_2 = \overrightarrow{OQ} \times \overrightarrow{OR} = 6\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$. And that $\hat{\boldsymbol{n}}_2 = (2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}})/\sqrt{5}$.

The angle θ between these vectors can be obtained from their dot product:

$$\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{n}}_2 = \cos\theta,$$

and so $\theta = \cos^{-1}(4/5) \approx 36.9°$.

(b) The normal to the face PQR is $\boldsymbol{n}_3 = \overrightarrow{PQ} \wedge \overrightarrow{PR}$, where

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (0,0,3) - (2,1,1) = (-2,-1,2),$$
$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (1,2,2) - (2,1,1) = (-1,1,1).$$

We obtain:

$$\overrightarrow{PQ} \wedge \overrightarrow{PR} = \boldsymbol{n}_3 = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ -1 & 1 & 1 \\ -2 & -1 & 2 \end{vmatrix} = 3\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{k}}$$

We can get the angle ϕ, once again from the dot product:

$$\begin{aligned} \overrightarrow{OP} \cdot \boldsymbol{n}_3 &= |\overrightarrow{OP}||\boldsymbol{n}_3| \cos\phi, \\ (2,1,1) \cdot (3,0,3) &= \sqrt{4+1+1}\sqrt{3^2+3^2}\cos\phi, \\ 9 &= 6\sqrt{3}\cos\phi, \end{aligned}$$

and so $\phi = \cos^{-1}(\sqrt{3}/2) = \pi/6$.

(c) The area of the triangle PQR may be expressed using the vector product

$$= \frac{1}{2}|\overrightarrow{PR} \wedge \overrightarrow{PQ}| = \frac{1}{2}|3\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{k}}| = \frac{3\sqrt{2}}{2}.$$

(d) Let us first derive the equation of the plane defined by the triangular surface PQR. We require one point on the plane and the normal to the plane. We can select $\overrightarrow{OP} = (2, 1, 1) \equiv \boldsymbol{p}$ to be our point on the plane. A normal to the plane is actually just $\boldsymbol{n}_3$ calculated earlier, which we can normalise by $3\sqrt{2}$ to make a unit vector $\hat{\boldsymbol{n}}_3$. Then our equation of the plane is just

$$(\boldsymbol{r} - \boldsymbol{p}) \cdot \hat{\boldsymbol{n}}_3 = 0.$$

From earlier in the notes, another equation for the plane is $\boldsymbol{r} \cdot \hat{\boldsymbol{n}}_3 = d$, where d is the the closest point on the plane to the origin. By comparing the two equations we get

$$d = \boldsymbol{p} \cdot \hat{\boldsymbol{n}}_3 = (2, 1, 1) \cdot (1/\sqrt{2}, 0, 1/\sqrt{2}) = 3/\sqrt{2},$$

our answer.

Example 1.20 [Tripos question 2003 Paper 1, 1A]

- The vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ form reciprocal sets, defined such that

$$\boldsymbol{A} = \frac{\boldsymbol{b} \times \boldsymbol{c}}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})}, \quad \boldsymbol{B} = \frac{\boldsymbol{c} \times \boldsymbol{a}}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})}, \quad \boldsymbol{C} = \frac{\boldsymbol{a} \times \boldsymbol{b}}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})} \tag{84}$$

and $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) \neq 0$. Show that:

(a) $\boldsymbol{A} \cdot \mathrm{a} = \boldsymbol{B} \cdot \boldsymbol{b} = \boldsymbol{C} \cdot \boldsymbol{c} = 1$

(b) $\boldsymbol{A} \cdot \boldsymbol{b} = \boldsymbol{A} \cdot \boldsymbol{c} = 0$

(c) $\boldsymbol{A},\ \boldsymbol{B},\ \boldsymbol{C}$ are non-coplanar.

- Demonstrate that the vectors $\boldsymbol{a} = (-1, 1, 0)$ and $\boldsymbol{b} = (0, 2, 1)$ and $\boldsymbol{c} = (1, 0, -1)$ are non-coplanar. Find its reciprocal basis, $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$, and hence write the vector $\boldsymbol{d} = (2, 1, -1)$ in terms of the basis $\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c}$.

(a) Recall that

$$\boxed{\begin{aligned} \boldsymbol{a}\cdot\boldsymbol{b}\wedge\boldsymbol{c} &= \ \boldsymbol{b}\cdot\boldsymbol{c}\wedge\boldsymbol{a} \ = \boldsymbol{c}\cdot\boldsymbol{a}\wedge\boldsymbol{b} \\ &= -\boldsymbol{a}\cdot\boldsymbol{c}\wedge\boldsymbol{b} = -\boldsymbol{b}\cdot\boldsymbol{a}\wedge\boldsymbol{c} = \ -\boldsymbol{c}\cdot\boldsymbol{b}\wedge\boldsymbol{a} \end{aligned}}. \tag{85}$$

Thus,

$$\begin{aligned} \boldsymbol{A}\cdot\boldsymbol{a} &= \frac{\boldsymbol{b}\wedge\boldsymbol{c}\cdot\boldsymbol{a}}{\boldsymbol{a}\cdot\boldsymbol{b}\wedge\boldsymbol{c}} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}\wedge\boldsymbol{c}}{\boldsymbol{a}\cdot\boldsymbol{b}\wedge\boldsymbol{c}} = 1, \\ \boldsymbol{B}\cdot\boldsymbol{b} &= \frac{\boldsymbol{c}\wedge\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{a}\cdot\boldsymbol{b}\wedge\boldsymbol{c}} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}\wedge\boldsymbol{c}}{\boldsymbol{a}\cdot\boldsymbol{b}\wedge\boldsymbol{c}} = 1, \\ \boldsymbol{C}\cdot\boldsymbol{c} &= \frac{\boldsymbol{a}\wedge\boldsymbol{b}\cdot\boldsymbol{c}}{\boldsymbol{a}\cdot\boldsymbol{b}\wedge\boldsymbol{c}} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}\wedge\boldsymbol{c}}{\boldsymbol{a}\cdot\boldsymbol{b}\wedge\boldsymbol{c}} = 1. \end{aligned}$$

(b)

$$\begin{aligned} \boldsymbol{A}\cdot\boldsymbol{b} &= \frac{\boldsymbol{b}\wedge\boldsymbol{c}\cdot\boldsymbol{b}}{\boldsymbol{a}\cdot\boldsymbol{b}\wedge\boldsymbol{c}} = \frac{\boldsymbol{b}\wedge\boldsymbol{b}\cdot\boldsymbol{c}}{\boldsymbol{a}\cdot\boldsymbol{b}\wedge\boldsymbol{c}} = 0, \\ \boldsymbol{A}\cdot\boldsymbol{c} &= \frac{\boldsymbol{b}\wedge\boldsymbol{c}\cdot\boldsymbol{c}}{\boldsymbol{a}\cdot\boldsymbol{b}\wedge\boldsymbol{c}} = 0 \end{aligned}$$

(c) If $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are non-coplanar then their scalar triple product must be non-zero, i.e. $\boldsymbol{A} \cdot \boldsymbol{B} \wedge \boldsymbol{C} \neq 0$. Let us evaluate the triple product, but simplifying the cross product first:

$$\boldsymbol{B} \wedge \boldsymbol{C} = \frac{(\boldsymbol{c} \wedge \boldsymbol{a}) \wedge (\boldsymbol{a} \wedge \boldsymbol{b})}{(\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c})^2}.$$

Like in a previous worked example, the quadruple product can be simplified if we set $\boldsymbol{s} = \boldsymbol{c} \wedge \boldsymbol{a}$ and use the identity

$$\begin{aligned} \boldsymbol{s} \wedge (\boldsymbol{a} \wedge \boldsymbol{b}) &= (\boldsymbol{b} \cdot \boldsymbol{s})\boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{s})\boldsymbol{b}, \\ &= (\boldsymbol{b} \cdot \boldsymbol{c} \wedge \boldsymbol{a})\boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{c} \wedge \boldsymbol{c})\boldsymbol{b}, \\ &= (\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c})\boldsymbol{a}. \end{aligned}$$

Thus $\boldsymbol{B} \wedge \boldsymbol{C} = (\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c})^{-1}\boldsymbol{a}$. Now let us look at the scalar triple product itself:

$$\boldsymbol{A} \cdot \boldsymbol{B} \wedge \boldsymbol{C} = \frac{(\boldsymbol{b} \wedge \boldsymbol{c}) \cdot \boldsymbol{a}}{(\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c})^2}, \\ = \frac{1}{\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c}} \neq 0$$

because $\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c} \neq 0$. Hence $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are *not* coplanar.

To show that $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are non-coplanar we look at

$$\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c} = \begin{vmatrix} -1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & -1 \end{vmatrix}, \\ = -1 \cdot \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} + 0, \\ = 2 + 1 = 3 \neq 0$$

so they are non-coplanar.

To compute the reciprocal basis we need some vector products:

$$\boldsymbol{b} \wedge \boldsymbol{c} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 0 & 2 & 1 \\ 1 & 0 & -1 \end{vmatrix} = (-2, 1, -2),$$

$$\boldsymbol{c} \wedge \boldsymbol{a} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{vmatrix} = (1, 1, 1),$$

$$\boldsymbol{a} \wedge \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ -1 & 1 & 0 \\ 0 & 2 & 1 \end{vmatrix} = (1, 1, -2),$$

And since their scalar triple product is 3, we have

$$\boldsymbol{A} = \tfrac{1}{3}(-2, 1, -2), \quad \boldsymbol{B} = \tfrac{1}{3}(1, 1, 1), \quad \boldsymbol{C} = \tfrac{1}{3}(1, 1, -2)$$

Now to write $\boldsymbol{d}$ in terms of the *non-orthogonal* basis $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ we could set

$$\boldsymbol{d} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c}$$

where α, β, and γ are constant components that we need to find. The x, y, and z coordinates of this vector equation yield three scalar equations for three unknowns, and it's easy to solve:

$$2 = -\alpha + \gamma, \quad 1 = \alpha + 2\beta, \quad -1 = \beta - \gamma,$$

for α, β, and γ.

A better way, using the reciprocal basis, is to dot both sides of the equation with $\boldsymbol{A}$. Using our previous results we obtain

$$\alpha = \boldsymbol{d} \cdot \boldsymbol{A} = (2, 1, -1) \cdot \tfrac{1}{3}(-2, 1, -2) = -\tfrac{1}{3},$$

similarly

$$\beta = \boldsymbol{d} \cdot \boldsymbol{B} = (2, 1, -1) \cdot \tfrac{1}{3}(1, 1, 1) = \tfrac{2}{3},$$
$$\gamma = \boldsymbol{d} \cdot \boldsymbol{C} = (2, 1, -1) \cdot \tfrac{1}{3}(1, 1, -2) = \tfrac{5}{3}.$$

Thus

$$\boldsymbol{d} = -\tfrac{1}{3}\boldsymbol{a} + \tfrac{2}{3}\boldsymbol{b} + \tfrac{5}{3}\boldsymbol{c}.$$

2 Complex numbers

Though they don't exist, *per se*, complex numbers pop up all over the place when we do science. They are an especially useful mathematical tool in problems involving:

- oscillations and waves in light, fluids, magnetic fields, electrical circuits, etc.,
- stability problems in fluid flow and structural engineering,
- signal processing (the Fourier transform, etc.),
- quantum physics, e.g. Schroedinger's equation,

  $$i\frac{\partial\psi}{\partial t} + \Delta\psi + V(x)\psi = 0\,,$$

  which describes the wavefunctions of atomic and molecular systems,
- difficult differential equations.

For your revision: recall that the general quadratic equation, $ax^2 + bx + c = 0$ (solving for x), has two solutions given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \,. \tag{86}$$

But what happens when the discriminant is negative?

For instance, consider

$$x^2 - 2x + 2 = 0 \,. \tag{87}$$

Its solutions are

$$\begin{aligned} x &= \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times 2}}{2 \times 1} = \frac{2 \pm \sqrt{-4}}{2} \\ &= 1 \pm \sqrt{-1} \,. \end{aligned}$$

These two solutions are evidently *not* ‘normal’ numbers!

If we put that aside, however, and accept the existence of the square root of minus one, written as

$$i = \sqrt{-1}\,, \tag{88}$$

then these two solutions, $1+i$ and $1-i$, can be assigned to the set of *complex numbers*. The quadratic equation now can be factorised as

$$z^2 - 2z + 2 = (z-1-i)(z-1+i) = 0\,,$$

and to be able to factorise every polynomial is very useful, as we shall see.

2.1 Complex algebra

2.1.1 Definitions

A complex number z takes the form

$$\boxed{z = x + iy\ ,} \tag{89}$$

where x and y are real numbers and i is the imaginary unit satisfying

$$i^2 = -1 \,. \tag{90}$$

We call x and y the real and imaginary parts of z respectively, and write

$$\boxed{\begin{array}{rclcll} x & = & \Re(z) & \text{or} & \mathrm{Re}(z) & \text{the real part of } z \,, \\ y & = & \Im(z) & \text{or} & \mathrm{Im}(z) & \text{the imaginary part of } z \,. \end{array}}$$

If two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal, i.e. $z_1 = z_2$, then their real and imaginary parts must be equal, that is, both $x_1 = x_2$ and $y_1 = y_2$.

2.1.2 Addition

The sum of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is also a complex number given by

$$\boxed{z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2) \,.} \tag{91}$$

The real part is $\Re(z_1 + z_2) = x_1 + x_2$ and the imaginary part is $\Im(z_1 + z_2) = y_1 + y_2$. The commutativity and associativity of real numbers under addition is therefore also passed on to the complex numbers, e.g. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.

2.1.3 Multiplication

We can multiply complex numbers, provided we know how to multiply i by itself.

The following table gives the results for the powers of i:

$$\boxed{\begin{array}{lllll} i & = & i & & \\ i^2 & = & -1 & & \\ i^3 & = & i^2 \times i & = & -i \\ i^4 & = & i^3 \times i & = & -i \times i = 1 \\ i^5 & = & i^4 \times i & = & i \\ \vdots & & \vdots & & \vdots \end{array}} \tag{92}$$

Note the pattern has a fourfold periodicity with $i^{4n+m} = i^m$, where n is any integer.

Okay, now consider the product of the two complex numbers $z_1 = x_1 + iy_1$ and

$z_2 = x_2 + iy_2$:

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 + i x_1 y_2 + i y_1 x_2 + i^2 y_1 y_2 \\ &= x_1 x_2 + i(x_1 y_2 + y_1 x_2) - y_1 y_2 \end{aligned}$$

where we have used (90). So, collecting real and imaginary parts, we have

$$\boxed{z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)\,.} \tag{93}$$

Complex multiplication inherits commutativity and associativity from the real numbers,

$$z_1 z_2 = z_2 z_1\,, \qquad z_1(z_2 z_3) = (z_1 z_2) z_3\,, \tag{94}$$

and it is also distributive

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3\,. \tag{95}$$

Example 2.1 If $z_1 = 3 + i$ and $z_2 = 1 - i$ calculate $z_1 + z_2$, $z_1 - z_2$ and $z_1 z_2$.

$$z_1 + z_2 = 3 + i + 1 - i = 4,$$

$$z_1 - z_2 = 3 + i - (1 - i) = 3 - 1 + (1 + 1)i = 2 + 2i,$$

$$\begin{aligned} z_1 z_2 = (3 + i)(1 - i) = 3 - 3i + i + i(-i) &= 3 - 2i - i^2, \\ &= 3 - 2i + 1, \\ &= 4 - 2i. \end{aligned}$$

2.1.4 Complex conjugate and modulus

The complex conjugate of $z = x + iy$ is found by changing the sign of its imaginary component. It is denoted by z^* (or often $\overline{z}$) and defined by

$$\boxed{z^* = x - iy\ .} \tag{96}$$

Note that it follows that we must have $z + z^* = 2\Re(z)$ and $z - z^* = 2i\Im(z)$. If we take the product of z with its complex conjugate z^* we find

$$zz^* = (x + iy)(x - iy) = x^2 + y^2 + i(-xy + xy) = x^2 + y^2\ ,$$

which is real and non-negative. Thus

$$\boxed{zz^* = x^2 + y^2\ .} \tag{97}$$

The *modulus* of z, denoted by $|z|$ or mod(z), is defined by

$$\boxed{|z| = \sqrt{zz^*} = \sqrt{x^2 + y^2}\ .} \tag{98}$$

2.1.5 Division

It is easiest to compute the division of one complex number by another by using the properties of the conjugate and modulus.

The division of z_1 by z_2 may be manipulated into

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{z_2^*}{z_2^*} = \frac{z_1 z_2^*}{|z_2|^2}, \tag{99}$$

and the denominator is conveniently a real number. Then we use the rule for multiplication of complex numbers, in the numerator, to compute the result.

The general rule for simplifying expressions involving division by a complex number z_2 is to multiply numerator and denominator by the complex conjugate z_2^*/z_2^* (i.e. the identity), thus making the denominator real $z_2 z_2^* = |z_2|^2$.

<u>Example 2.2</u> Take $z_1 = 3 + i$ and $z_2 = 1 - i$ again and calculate z_1/z_2.

$$\frac{z_1}{z_2} = \frac{3+i}{1-i} \times \frac{1+i}{1+i} = \frac{3+3i+i+i^2}{1^2+1^2} = 1+2i.$$

2.2 The Complex Plane

2.2.1 Argand diagram

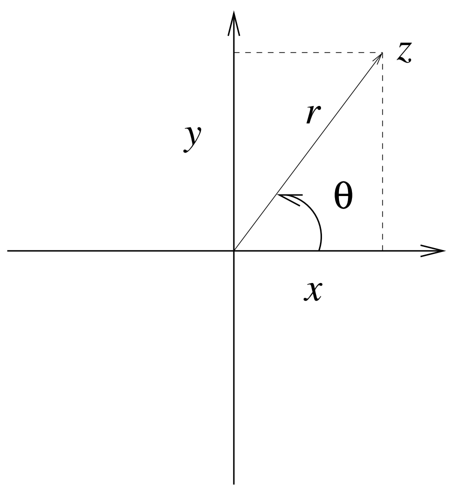


Figure 34: The Argand diagram of the complex plane.

The real (x) and imaginary (y) parts of a complex number z are independent quantities, so we can think of z as plotting a point in a two-dimensional space, (x, y), where the y axis corresponds to the imaginary part and the x axis corresponds to the real part of the number.

In fact, we can go further and think of a complex number as a two-dimensional vector.

This two-dimensional space is often called the Argand diagram.

Now purely algebraic processes (conjugation, addition, division, etc.) can be represented as geometric operations. For instance

- The complex conjugate z^* is found by merely reflecting any point z about the real (i.e. x) axis.

- Addition and subtraction of complex numbers is essentially equivalent to vector addition and subtraction.

2.2.2 Polar form of complex numbers

We can also use plane polar coordinates in the Argand plane. Simple trigonometry gives us

$$\boxed{r = \sqrt{x^2 + y^2} = |z|\,.} \tag{100}$$

The radius r is, in fact, the modulus of z, met earlier.

The polar angle θ is called the *argument* or *phase* of z. It is the angle subtended by the real axis and the line made by the complex number and the origin. The following formula gets us the argument

$$\boxed{\theta = \pm \cos^{-1}\left(\frac{x}{r}\right)\,,} \tag{101}$$

where the plus sign is taken when $y > 0$, and the minus sign when $y < 0$. (Another common formula is $\theta = \tan^{-1}(y/x)$, though on its own it fails to distinguish between the 1st and 3rd quadrant, or between the 2nd and 4th.)

It is conventional to restrict the argument θ to the range $-\pi < \theta \leqslant \pi$ in order to make it unique; this is called the $principal\ range$, and then θ is the *principal argument*. More generally, we can add any integer multiple of 2π to θ and z is unchanged. This is because $\cos(\theta + 2\pi n) = \cos\theta$ and $\sin(\theta + 2\pi n) = \sin\theta$, for integer n.

The polar representation of z is:

$$\boxed{z = x + iy = r(\cos\theta + i\sin\theta)\,.} \tag{102}$$

For the complex conjugate, it is clear that the argument will become $-\theta$, i.e.,

$$z^* = x - iy = r(\cos\theta - i\sin\theta) = r\left[\cos(-\theta) + i\sin(-\theta)\right]\ . \tag{103}$$

The inverse, on the other hand, is

$$z^{-1} = \frac{z^*}{|z|^2} = r^{-1}\left[\cos(-\theta) + i\sin(-\theta)\right] \tag{104}$$

Example 2.3 Calculate the modulus and argument for $z_1 = 3 + i$ and $z_2 = 1 - i$.

For z_1, we have $r = |z_1| = \sqrt{3^2 + 1^2} = \sqrt{10}$, while $\theta = +\cos^{-1}(3/\sqrt{10}) \approx 18.43^\circ$, where we take the positive sign because the imaginary part of z_1 is positive.

For z_2, $r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$, and $\theta = -\cos^{-1}(1/\sqrt{2}) = -\pi/4$, where we have taken the negative sign because the imaginary part of z_2 is negative

Example 2.4 What shape in the Argand diagram is described by the equation $3|z| = |z - i|$? What about $|z| = |z - i|$?

First set $z = x + iy$. Squaring the first equality yields

$$9|z|^2 = |z - i|^2,$$
$$9(x^2 + y^2) = |x + i(y - 1)|^2 = x^2 + (y - 1)^2 = x^2 + y^2 - 2y + 1,$$
$$\rightarrow 8x^2 + 8y^2 + 2y = 1,$$
$$x^2 + y^2 + \tfrac{1}{4}y = \tfrac{1}{8},$$
$$x^2 + (y + \tfrac{1}{8})^2 = (\tfrac{3}{8})^2$$

So in the Argand diagram we have a circle centred at $(0, -\frac{1}{8})$ (or $z = -i/8$) with radius $\frac{3}{8}$.

Let us also square the second equality:

$$|z|^2 = |z - i|^2$$
$$x^2 + y^2 = x^2 + (y - 1)^2 = x^2 + y^2 - 2y + 1,$$
$$\rightarrow y = \tfrac{1}{2}.$$

Since x is arbitrary, $z = x + \frac{1}{2}i$, a straight line. Points on the line are equidistant from $z = 0$ and $z = i$.

2.3 The complex exponential

2.3.1 Euler's formula

There is a profound relationship between trigonometric functions and the complex exponential function:

$$\cos\theta + i\sin\theta = e^{i\theta} . \tag{105}$$

We will prove this later. This is called Euler's formula.

Euler's formula means that we can write any complex number compactly as

$$\boxed{z = r\, e^{i\theta} .} \tag{106}$$

As mentioned earlier, there is some degeneracy here because we can add any integer multiple of 2π onto θ without changing the value of z. An interesting consequence is the useful identity:

$$\exp(2i\pi n) = 1,$$

for integer n.

2.3.2 Multiplication

The exponential form makes it *easy* to do multiplication and division. If $z_1 = r_1 \exp(i\theta_1)$ and $z_2 = r_2 \exp(i\theta_2)$ then

$$\boxed{z_1 z_2 = r_1 r_2 \exp\left(i(\theta_1 + \theta_2)\right)\ .} \tag{107}$$

So when multiplying complex numbers the moduli are *multiplied* together and the arguments are *added*.

2.3.3 Division

We looked at division previously (99) and this now becomes

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \exp(i[\theta_1 - \theta_2]) \ . \tag{108}$$

So when dividing complex numbers the moduli are divided and the arguments are subtracted.

2.3.4 Geometric manifestations

A geometrical interpretation of multiplication of z_1 by z_2 corresponds to rotation of z_1 by the argument of z_2 and a scaling of z_1's modulus by $|z_2|$. Note the special case of multiplication by i corresponds simply to rotation by 90° anti-clockwise. Taking the complex conjugate of z corresponds to reflection in the x axis.

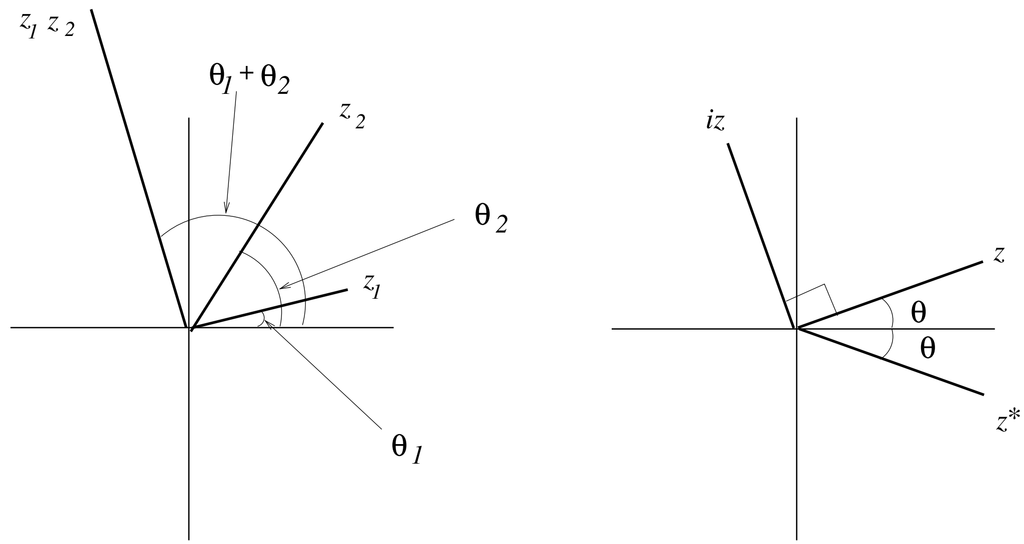


Figure 35: Geometrical interpretation of (a) multiplication of z_1 and z_2; (b) multiplication by i and complex conjugation.

2.3.5 New expressions for cos and sin

Taking the complex conjugate of (105) yields

$$\exp(-i\theta) = \cos\theta - i\sin\theta \ , \tag{109}$$

and adding (105) to (109) gets us

$$\boxed{\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \ .} \tag{110}$$

Similarly, subtracting (109) from (105) gives an expression for sin

$$\boxed{\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \ .} \tag{111}$$

2.3.6 Fundamental Theorem of Algebra

We state the fundamental theorem of algebra without proof. The polynomial equation of degree n (a positive integer)

$$a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0\,, \qquad a_n \neq 0\,. \tag{112}$$

has n complex roots for any possible (complex) coefficients $a_0, a_1, \ldots a_n$.

2.3.7 Roots of unity

We want to solve the equation

$$z^n = 1\,, \tag{113}$$

where n is an integer. One root is $z = 1$, of course, but (113) is a polynomial equation of degree n, so we expect n roots!

The way to grab these is to use the complex exponential notation and allow for degeneracy in the complex argument.

- We recognise that 1 is a complex number with modulus 1 and argument $0 + 2\pi m$, for m an integer:

$$z^n = 1 = \exp(2\pi i m) \qquad m = 0, 1, 2, \ldots\ldots . \tag{114}$$

- then take the nth root of both sides

$$z = \exp(2\pi i m/n) \qquad m = 0, 1, 2, \ldots \quad . \tag{115}$$

However, the root with $m = n$ is $z = \exp(2\pi i) = 1$, which is the same as the root with $m = 0$, so the n *distinct* roots of equation (113) are

$$\boxed{z = \exp(2\pi i m/n) \qquad m = 0, 1, 2, \ldots n - 1 \ .} \tag{116}$$

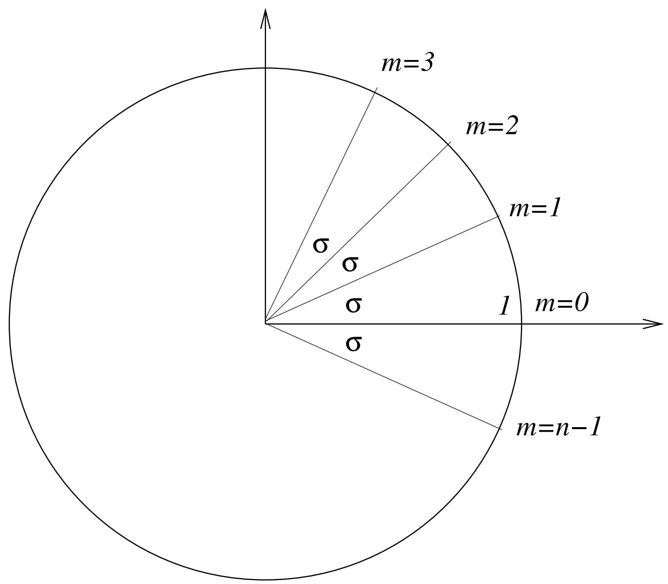


Figure 36: The roots of $z^n = 1$ in the complex plane, given by (116). The angle $\sigma = 2\pi/n$.

Alternatively, we can write $\omega = e^{2\pi i/n}$, and the n roots are $1, \omega, \omega^2, ..., \omega^{n-1}$. These roots are distributed around the unit circle (with radius 1 centred on the origin) at regular angles of $2\pi/n$.

<u>Example 2.5</u> Solve $z^5 = 2i$.

$$
\begin{aligned}
z^5 &= 2i = 2\exp\left(i\frac{\pi}{2} + 2i\pi n\right), \quad n = 0, 1, 2, \ldots \\
z &= 2^{1/5}\exp\left(i\frac{\pi}{10} + \frac{2i\pi n}{5}\right).
\end{aligned}
$$

Note that when $n = 5$ we repeat the root for $n = 0$, etc. So five roots are:

$$
\begin{aligned}
&2^{1/5}\exp\left(i\frac{\pi}{10}\right), \quad 2^{1/5}\exp\left(i\frac{\pi}{2}\right) = 2^{1/5}i, \\
&2^{1/5}\exp\left(i\frac{9\pi}{10}\right), \quad 2^{1/5}\exp\left(i\frac{13\pi}{10}\right) \quad 2^{1/5}\exp\left(i\frac{17\pi}{10}\right)
\end{aligned}
$$

2.4 De Moivre's Theorem

We can use the exponential form of a complex number to derive a very useful result for obtaining trigonometric identities.

First, recall

$$e^{i\theta} = \cos\theta + i\sin\theta\ . \tag{117}$$

We can replace θ by $n\theta$ and write

$$e^{in\theta} = \cos n\theta + i\sin n\theta\ . \tag{118}$$

Also, we know that

$$e^{in\theta} = [e^{i\theta}]^n\ . \tag{119}$$

Combining these results yields

$$\boxed{\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n\ .} \tag{120}$$

This is called *De Moivre's Theorem*. Note that n does not have to be an integer. De Moivre's Theorem, and the complex exponential more generally, are very useful indeed for working out expressions for multiple angle formulae, such as for $\cos 4\theta$ and $\sin 4\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$, and vice versa.

For instance, rewriting cos in terms of complex exponentials:

$$\begin{aligned}
\cos^3\theta &= \left(\tfrac{1}{2}\right)^3 [\exp(i\theta) + \exp(-i\theta)]^3 \\
&= \tfrac{1}{8}[\exp(3i\theta) + 3\exp(i\theta) + 3\exp(-i\theta) + \exp(-3i\theta)] \\
&= \tfrac{1}{8}(2\cos 3\theta + 6\cos\theta) \\
&= \tfrac{1}{4}\cos 3\theta + \tfrac{3}{4}\cos\theta\ .
\end{aligned} \tag{121}$$

Another application is to work out sums of trigonometric functions.

For example, if we wish to sum the series

$$\sum_{r=0}^{N} \cos r\theta\ ,$$

then the thing to do is to write

$$\sum_{r=0}^{N} \cos r\theta = \Re\left[\sum_{r=0}^{N} \exp(ir\theta)\right]\ . \tag{122}$$

The series

$$\sum_{r=0}^{N} \exp(ir\theta)$$

is actually a geometric progression with first term 1 and common ratio $\exp(\mathrm{i}\theta)$, for which we can write down the answer, and then the cosine series we want follows by taking the real part. We do this in detail in Example 2.7.

Example 2.6 Use De Moivre's Theorem to find expressions for $\cos 4\theta$ and $\sin 4\theta$.

$$\begin{aligned}\cos 4\theta + i\sin 4\theta &= (\cos\theta + i\sin\theta)^4, \\ &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\ &\qquad + 4i\cos^3\theta\sin\theta - 4i\cos\theta\sin^3\theta.\end{aligned}$$

Take real part of the equation:

$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta, \\ &= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2, \\ &= 8\cos^4\theta - 8\cos^2\theta + 1.\end{aligned}$$

Take imaginary part:

$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$$

Example 2.7 Evaluate $\sum_{k=0}^{N}\cos k\theta$.

First note that $\sum_{k=0}^{N}\cos k\theta = \mathrm{Re}\sum_{k=0}^{N}\exp(ik\theta)$. So what we have is a finite geometric sum: first term is 1, the common ratio is $\exp(i\theta)$. We can then use the formula for the geometric sum:

$$\sum_{k=0}^{N} \exp(ik\theta) = \frac{1 - \exp[i(N+1)\theta]}{1 - \exp(i\theta)},$$

$$= \frac{1 - \exp[i(N+1)\theta]}{1 - \exp(i\theta)} \cdot \frac{1 - \exp(-i\theta)}{1 - \exp(-i\theta)},$$

$$= \frac{1 - \exp(i\theta) - \exp[i(N+1)\theta] + \exp(iN\theta)}{2 - 2\cos\theta}.$$

Take the real part:

$$\sum_{k=0}^{N} \cos(k\theta) = \frac{1 - \cos\theta - \cos(N+1)\theta + \cos N\theta}{2(1 - \cos\theta)},$$

$$= \frac{1}{2} + \frac{\cos N\theta - \cos(N+1)\theta}{2(1 - \cos\theta)}.$$

2.5 Complex logarithms

Having discussed the complex exponential, the obvious thing to do next is to consider the inverse function, i.e. the complex natural logarithm $\ln z$. We first write z in the exponential form $z = |z| \exp(i\theta)$, and then:

$$\boxed{\begin{aligned} \ln z &= \ln(|z| \exp(i\theta)) \\ &= \ln(|z|) + \ln(\exp(i\theta)) \\ &= \ln(|z|) + i\theta \end{aligned}} \tag{123}$$

Of course, as we have already noted, the argument θ of z is really multi-valued, in the sense that we can add any integer multiple of 2π onto θ without changing the value of z. This means that $\ln z$ is a multi-valued function.

Often the *principal value* of $\ln z$ is defined by choosing just one of these possible values of θ, and the usual convention with $\ln z$ is to choose $-\pi < \theta < \pi$.

As an example, we work out $\ln(2i)$. First,

$$2i = 2\exp(i\pi/2 + 2n\pi i) \quad \text{for } n = 0, \pm 1, \pm 2, \dots\,, \tag{124}$$

and then using (123) we see that

$$\ln(2i) = \ln 2 + i\left(\frac{\pi}{2} + 2\pi n\right)\,. \tag{125}$$

Example 2.8 Re-express 2^i and i^i.

$$2^i = [\exp(\ln 2)]^i = [\exp(\ln 2 + 2\pi i n)]^i = \exp(-2\pi n)\exp(i\ln 2), \qquad n = 0, 1, 2, \dots$$

Taking $n = 0$ gives us the principal value $\exp(i\ln 2)$.

$$i^i = [\exp(i\pi/2 + 2\pi i n)]^i = \exp(-\pi/2)\exp(-2\pi n), \qquad n = 0, 1, 2, \dots.$$

Again setting $n = 0$ yields the nice principal value $e^{-\pi/2}$.

2.6 Oscillation problems

Complex numbers are especially useful in problems which involve oscillatory or periodic motion, such as when describing the motion of a simple pendulum, alternating electrical circuits, or any sort of wave motion in air and water.

To be specific, let us consider a simple pendulum swinging under gravity with angular frequency ω.

The angular displacement, $x(t)$, of the pendulum about the vertical then takes the general form

$$x(t) = a\cos\omega t + b\sin\omega t \ , \tag{126}$$

where a and b are real constants.

Using complex numbers we can write this as

$$x(t) = \Re\left[A\exp(i\omega t)\right] \ , \tag{127}$$

where A is a *complex* constant. In fact, by comparing (126) and (127) we find that

$$A = a - ib \; . \tag{128}$$

The big advantage of the complex representation (127) is that differentiation is very easy. For example, the velocity $v(t)$ is given by

$$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\Re\left[A\exp(i\omega t)\right] = \Re\left[i\omega A\exp(i\omega t)\right] \; . \tag{129}$$

In other words, to differentiate we simply multiply by $i\omega$. This idea leads to various transform methods of solving differential equations.

Example 2.9 A particle of mass m hangs from a metal spring with spring constant k. A driving force $F = F_0\cos(\omega t)$ is applied to the particle. Assuming the particle's vertical displacement $z(t)$ obeys the differential equation (Newton's second law):

$$m\frac{d^2z}{dt^2} + kz = F,$$

determine the amplitude of the resulting oscillation.

First assume a solution of the form $z = \Re[A\exp(i\omega t)]$, for some (possibly complex) amplitude A yet to be determined, and then re-express the forcing as $F = \Re[F_0\exp(i\omega t)]$. The differential equation is then transformed into the real part of the algebraic equation

$$(-m\omega^2 + k)A\exp(i\omega t) = F_0\exp(i\omega t),$$

and we solve for A directly:

$$A = \frac{F_0}{k - m\omega^2},$$

assuming that $\omega \neq \sqrt{k/m}$ (no resonance). Being real the particle oscillation is in phase with the forcing.

3 Hyperbolic Functions

So far you've met a small number of important and commonly used functions, such as $\sin(x)$, $\cos(x)$, $\tan(x)$, $\exp(x)$, and $\ln(x)$. But there is, in fact, an entire zoo of interesting functions that emerge when solving various physical problems (Bessel functions, spherical harmonics, hypergeometric functions, elliptic integrals, etc.). In this section, we will introduce a class of new functions, that are related to the usual circular functions (cos, sin, tan). These are the *hyperbolic functions*.

3.1 Definitions

The hyperbolic functions are denoted and defined through:

$$\boxed{\begin{aligned} \cosh x &= \tfrac{1}{2}\left(e^{x}+e^{-x}\right), \\ \sinh x &= \tfrac{1}{2}\left(e^{x}-e^{-x}\right), \\ \tanh x &= \sinh x/\cosh x = \tfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}. \end{aligned}} \tag{130}$$

and are pronounced 'cosh', 'shine', and 'tansh'. In the same way as with circular trigonometric functions, we also define

$$\boxed{\begin{aligned} \operatorname{sech} x &= 1/\cosh x \\ \operatorname{cosech} x &= 1/\sinh x \\ \coth x &= 1/\tanh x\ . \end{aligned}} \tag{131}$$

There is in fact a very close relationship between circular and hyperbolic trigonometric functions that involves complex numbers. Recall equation (110),

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \ .$$

Now consider

$$\cos(iz) = \frac{1}{2}\left[\exp(i^2 z) + \exp(-i^2 z)\right] = \frac{1}{2}\left(e^{-z} + e^{z}\right) \ ,$$

and comparing this result to equation (130) tells us that

$$\boxed{\cos(iz) = \cosh z \ .} \tag{132}$$

Similarly, recalling equation (111) we see that

$$\begin{aligned} \sin iz = \frac{1}{2i}\left[\exp(i^2 z) - \exp(-i^2 z]\right) &= \frac{1}{2i}\left(e^{-z} - e^{z}\right) \\ &= \frac{i}{2}\left(-e^{-z} + e^{z}\right) \ , \end{aligned}$$

so that

$$\boxed{\sin(iz) = i \sinh z \ .} \tag{133}$$

Dividing (133) by (132) gives

$$\boxed{\tan(iz) = i \tanh z\ .} \tag{134}$$

3.2 Identities

All the identities we know for circular trigonometric functions can be converted to corresponding identities for hyperbolic functions.

For example, consider $\cos^2 x + \sin^2 x = 1$. For hyperbolic functions, take

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= \frac{1}{4}[\exp(x) + \exp(-x)]^2 - \frac{1}{4}[\exp(x) - \exp(-x)]^2 \\
&= \frac{1}{4}[\exp(2x) + \exp(-2x) + 2] - \frac{1}{4}[\exp(2x) + \exp(-2x) - 2] \\
&= \frac{1}{4}(4)\ ,
\end{aligned}$$

so that

$$\boxed{\cosh^2 x - \sinh^2 x = 1}\ . \tag{135}$$

Note the different sign compared to the classical trig relationship!

To derive multi-angle formulas we exploit the relationships between 'cosh' and 'cos' etc. For instance, start with the trigonometric identity

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \ .$$

This identity is actually true for all *complex* A and B, not just real values, so we could equally well have

$$\cos(iA + iB) = \cos iA \cos iB - \sin iA \sin iB \ ,$$

and then using (132) & (133) we find that

$$\cos(iA + iB) = \cosh(A + B) = \cosh A \cosh B - i^2 \sinh A \sinh B \ ,$$

so that

$$\boxed{\cosh(A + B) = \cosh A \cosh B + \sinh A \sinh B} \ . \tag{136}$$

Here are some more identities:

$$\begin{aligned}
\sinh(A+B) &= \sinh A \cosh B + \sinh B \cosh A \\
\sinh(A-B) &= \sinh A \cosh B - \sinh B \cosh A \\
\cosh(A-B) &= \cosh A \cosh B - \sinh A \sinh B \quad . \\
1 - \tanh^2 z &= \mathrm{sech}^2 z \\
\coth^2 z - 1 &= \mathrm{cosech}^2 z
\end{aligned}$$

<u>Example 3.1</u> Find an identity for $\tanh(A+B)$.

$$
\begin{aligned}
\tanh(A+B) &= \frac{\sinh(A+B)}{\cosh(A+B)}, \\
&= \frac{\sin(iA+iB)}{i\cos(iA+iB)}, \\
&= \frac{\sin(iA)\cos(iB)+\cos(iA)\sin(iB)}{i[\cos(iA)\cos(iB)-\sin(iA)\sin(iB)]}, \\
&= \frac{i\sinh A\cosh B + i\cosh A\sinh B}{i(\cosh A\cosh B - i^2\sinh A\sinh B)}, \\
&= \frac{\sinh A\cosh B+\cosh A\sinh B}{\cosh A\cosh B+\sinh A\sinh B}, \\
&= \frac{\tanh A+\tanh B}{1+\tanh A\tanh B},
\end{aligned}
$$

where in the last line we divided both numerator and denominator by $\cosh A\cosh B$.

3.3 Graphs of hyperbolic functions

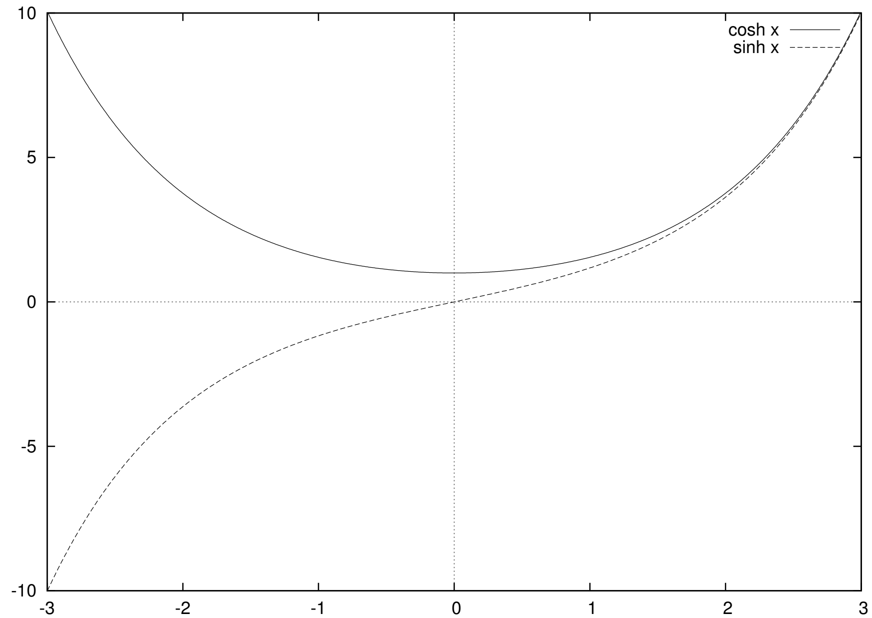


Figure 37: Plots of $\cosh x$ and $\sinh x$.

When sketching the functions, note the following:

- $\cosh 0 = 1$, $\sinh 0 = \tanh 0 = 0$.

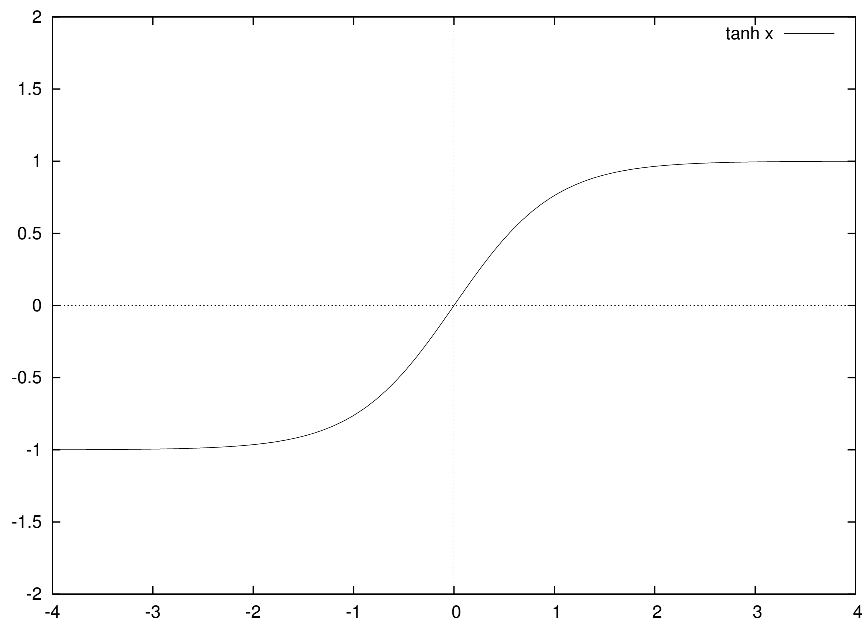


Figure 38: Plot of $\tanh x$. This is often used to represent a smooth step function.

- The graph of $\cosh x$ is symmetric about $x = 0$, i.e. $\cosh x = \cosh(-x)$, while $\sinh x$ and $\tanh x$ are antisymmetric,

- As x approaches infinity through positive values, $y = \cosh x$ and $y = \sinh x$

approach the same curve $\frac{1}{2}\exp(x)$. This is because as x gets bigger $\exp(-x)$ is much less than $\exp(x)$,

- As x gets more and more negative, the term $\exp(-x)$ dominates. Hence $\sinh x$ approaches $-\frac{1}{2}\exp(-x)$, and $\cosh x$ approaches $\frac{1}{2}\exp(-x)$
- The previous two points explain why $\tanh x \to \pm 1$ as $x \to \pm\infty$.
- Note that $\cosh x \geqslant 1$ and $-1 < \tanh x < 1$ for all real x.

3.4 Inverse hyperbolic functions

Having defined the hyperbolic functions, we now want to introduce their inverses. Consider first the inverse of $\sinh x$, denoted

$$y = \sinh^{-1} x \ ,$$

which means

$$\sinh y = x \ . \tag{137}$$

Using the definition of sinh gives

$$\frac{1}{2}\left(e^{y} - e^{-y}\right) = x \ , \tag{138}$$

and multiplying through by $2e^{y}$ gives

$$e^{2y} - 2xe^{y} - 1 = 0 \ , \tag{139}$$

a quadratic equation in $\exp y$. The solutions are

$$e^{y} = x \pm \sqrt{x^2 + 1} \ . \tag{140}$$

Of these two roots, the one with the minus sign is negative, and must be thrown away (the exponential of a real number can never be negative).

Taking the log of the remaining root yields

$$\boxed{y = \sinh^{-1} x \equiv \ln\left(x + \sqrt{x^2 + 1}\right)} \ . \tag{141}$$

To obtain inverse cosh, set $y = \cosh^{-1} x$, and after almost the same algebra, we arrive at the equation

$$e^{2y} - 2xe^{y} + 1 = 0 \ , \tag{142}$$

with the two roots

$$e^{y} = x \pm \sqrt{x^2 - 1} \ . \tag{143}$$

Both roots are positive, so both must be kept.

Taking logs

$$y = \cosh^{-1} x \equiv \ln\left(x \pm \sqrt{x^2 - 1}\right) \ ,$$

our answer.

But the $\pm$ can be brought outside the log. First note that

$$x - \sqrt{x^2 - 1} = \frac{1}{x + \sqrt{x^2 - 1}} \ , \tag{144}$$

so that

$$\ln(x - \sqrt{x^2 - 1}) = \ln\left[\frac{1}{x + \sqrt{x^2 - 1}}\right] = -\ln(x + \sqrt{x^2 - 1}) \ .$$

We are therefore finally left with the answer

$$\boxed{y = \cosh^{-1} x \equiv \pm\ln\left(x + \sqrt{x^2 - 1}\right) \ .} \tag{145}$$

Obviously, $\cosh^{-1}$ is not defined when $x < 1$. That is because $\cosh(x) \geq 1$.

The reason that the inverse cosh is multivalued (i.e. possesses the $\pm$) is that cosh is symmetric ($\cosh x = \cosh(-x)$). So for every possible value of y on the $y = \cosh x$ curve there is a positive *and* a negative value of x.

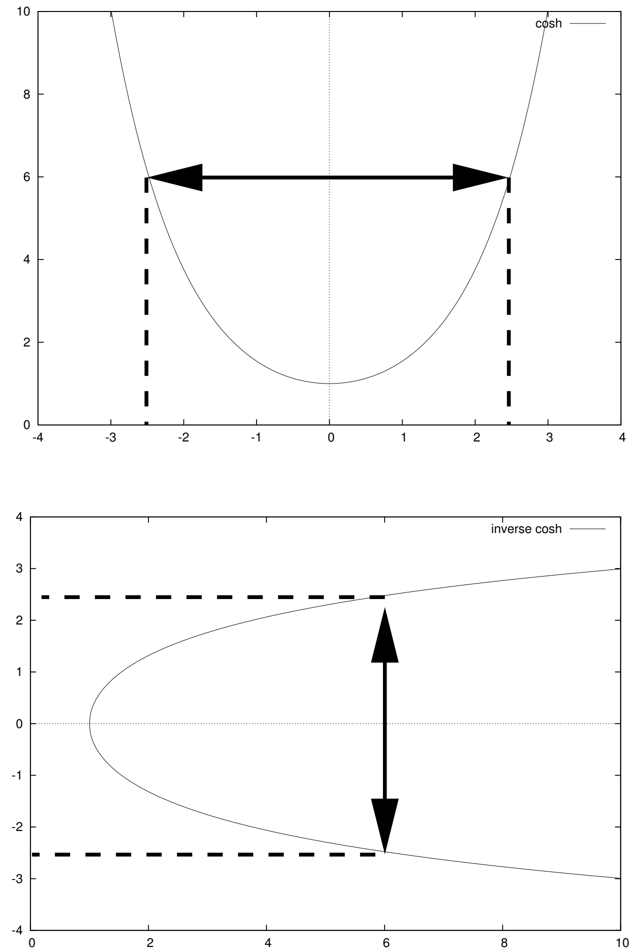


Figure 39: Plot of (a) $y = \cosh x$ and (b) $y = \cosh^{-1} x$.

Example 3.2 Find an identity for $\tanh^{-1} x$.

$$x = \tanh y = \frac{\sinh y}{\cosh y} = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y} - 1}{e^{2y} + 1}.$$

We now solve this equation for e^{2y}. Rearranging we get

$$x(e^{2y} + 1) = e^{2y} - 1,$$
$$\to e^{2y}(x - 1) = -x - 1,$$
$$\to e^{2y} = \frac{1 + x}{1 - x},$$

and so finally:

$$y = \tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1 + x}{1 - x} \right).$$

Example 3.3 Find all the roots of $\cos z = 2$.

Obviously we are dealing with a complex root, because $\cos$ takes values between

-1 and 1 for real arguments. Let $z = x + iy$, and so

$$\cos(x + iy) = \cos x \cos(iy) - \sin x \sin(iy) = 2,$$
$$\rightarrow \cos x \cosh y - i \sin x \sinh y = 2.$$

Take the real and imaginary parts

$$\cos x \cosh y = 2, \qquad -\sin x \sinh y = 0,$$

respectively. Let us solve the imaginary (second) part first. There are two possibilities:

1. $\sinh y = 0$, and so $y = 0$. Returning then to the real part of our original equation we have $\cos x = 2$, which is impossible because x has to be real. So this branch is no good.

2. $\sin x = 0$, and so $x = n\pi$, for $n = 0, 1, 2, \ldots$. Let us take n odd and go to the real part of the original equation, which becomes $-\cosh y = 2$. But then

there are no real solutions for y (and y has to be real). So this is no good either.

Let us finally take n even. The real part of the equation is now $\cosh y = 2$, which *can* be solved, and so $y = \cosh^{-1} 2 = \pm \ln(2 + \sqrt{3})$.

In summary there are an infinite number of complex solutions to the equation:

$$z = 2m\pi \pm i \ln(2 + \sqrt{3}), \qquad m = 0, 1, 2, \ldots$$

3.5 Circles, ellipses, and hyperbolae

3.5.1 Circles

As you all know, a circle in the xy plane with centre at the origin and with radius a has equation:

$$x^2 + y^2 = a^2. \tag{146}$$

But the curve can also be represented parametrically via

$$x = a\cos\theta, \qquad y = a\sin\theta, \tag{147}$$

where θ is the polar angle.

3.5.2 Ellipses

One way to generate an ellipse is to take a circle and then stretch one or both of the x and y axes. We then obtain the equation of the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{148}$$

where now a is called the *semi-major axis*, and b is the *semi-minor* axis. An ellipse need not have its semi-major and semi-minor axes aligned with the x and y axes, but in the canonical form above this is the case.

The curve has the parametric representation:

$$x = a\cos\theta, \qquad y = b\sin\theta, \tag{149}$$

where again θ is the polar angle.

An important quantity is the *eccentricity* $e = \sqrt{1 - b^2/a^2}$, which measures the degree of the ellipse's 'distortion'.

3.5.3 Hyperbolae

The equation of a hyperbola centred on the origin and aligned with the x and y axes is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{150}$$

Its parametric representation is

$$x = \pm a\cosh\theta, \qquad y = b\sinh\theta, \tag{151}$$

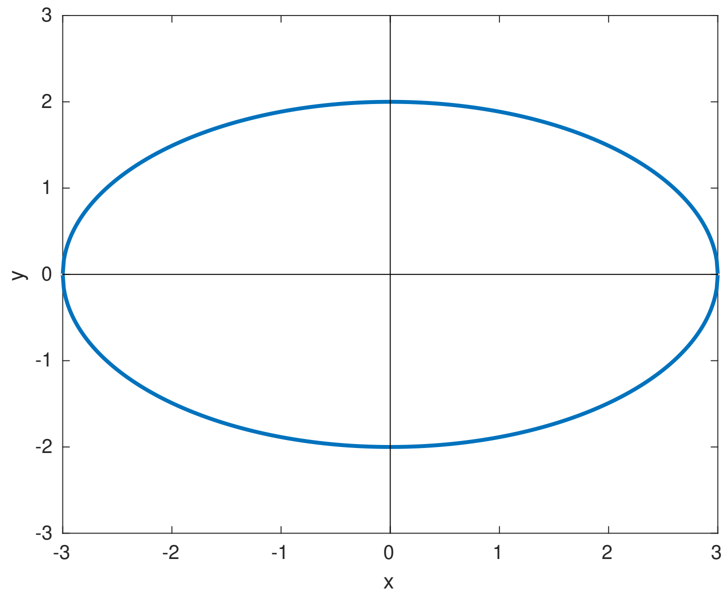


Figure 40: An ellipse with semi-major axis of $a = 3$ and semi-minor axis of $b = 2$.

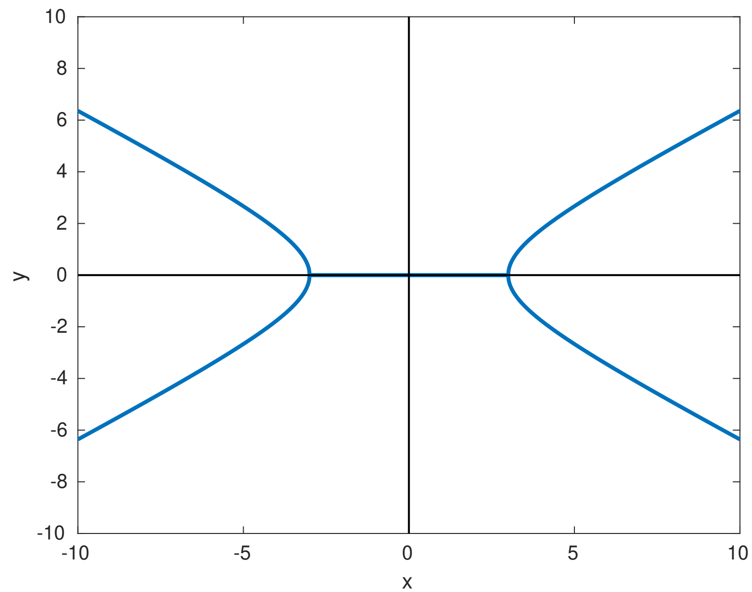


Figure 41: A hyperbola with semi-major axis of $a = 3$ and semi-minor axis of $b = 2$.

4 Differentiation

Rates of change, usually with respect to time and space, underpin so many of our scientific theories of the world. They regularly appear in the governing *differential* equations of a theory. There are almost too many examples to list, but prominent equations include: Newton's second law (dynamics), Schroedinger's equation (quantum mechanics), Einstein's field equations (general relativity), the Navier-Stokes equation (fluid dynamics), the Malthus and logistic models (population growth), Fisher's equation (gene propagation), and chemical reaction kinetics (chemistry).

In this part of the course we revise the basics of *differentiation*, which provides the mathematical foundations of change. We focus only on functions of a single variable.

4.1 First Principles

The derivative of a function $y(x)$ at a given point x will be denoted dy/dx and is defined by the limiting process:

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} \equiv \lim_{\delta x \to 0} \frac{y(x + \delta x) - y(x)}{\delta x} \ .} \tag{152}$$

Geometrically, the derivative is the gradient of the tangent line to the curve given by $y(x)$ at the point x. The tangent line has a slope such that it only just touches the curve at this point.

Example: differentiate $y = x^3$ from first principles:

$$\begin{aligned} y(x + \delta x) &= (x + \delta x)^3 = x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 \\ y(x + \delta x) - y(x) &= 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 \ . \end{aligned} \tag{153}$$

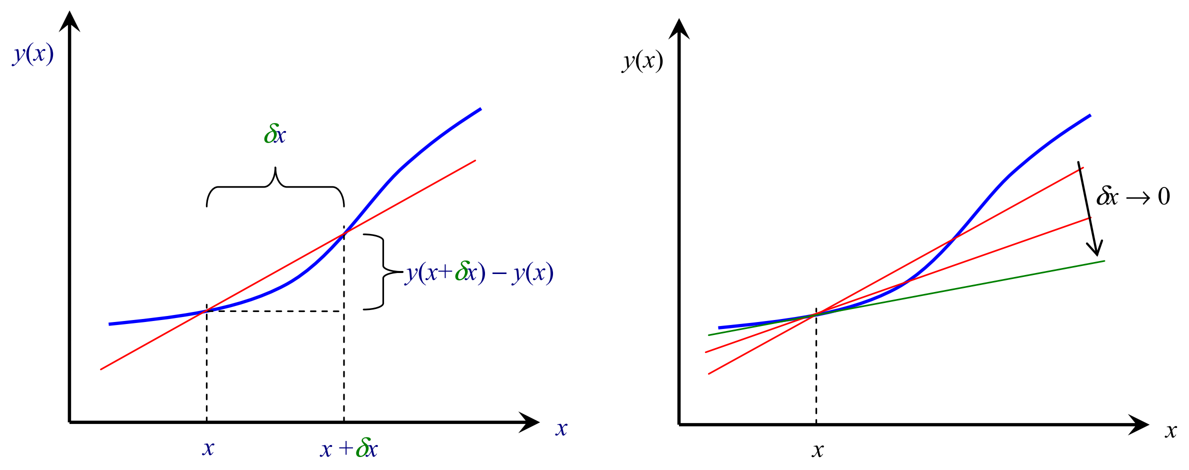


Figure 42: The derivative understood as the gradient of a secant in the limit of its two points merging.

Now

$$\frac{y(x+\delta x) - y(x)}{\delta x} = 3x^2 + 3x(\delta x) + (\delta x)^2 \ .$$

Take the limit $\delta x \to 0$ and the second and third terms on the right disappear, so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} \equiv \lim_{\delta x \to 0} \frac{y(x+\delta x) - y(x)}{\delta x} = 3x^2 \ . \tag{154}$$

4.1.1 Differentiability

Functions are not necessarily differentiable everywhere.

Example 1: consider the *Heaviside step function* $\mathrm{H}(x)$, defined so that $\mathrm{H}(x) = 0$ for $x < 0$ and $\mathrm{H}(x) = 1$ for $x \geq 0$. What is its derivative at $x = 0$?

If we approach the limit in the definition of the derivative from negative values of δx, then $[y(0) - y(\delta x)]/\delta x = 1/\delta x$, which diverges as $\delta x \to 0$. So the derivative at $x = 0$ using (152) cannot exist.

This is an example of a *discontinuous function*; such functions are not differentiable at their discontinuities.

Example 2: consider the absolute value function $y(x) = |x|$, which is continuous but *not smooth* at $x = 0$.

At $x = 0$, using the formal definition of the derivative, we obtain $d|x|/dx = 1$ if we approach the limit from above $x = 0$ (positive δx), and $d|x|/dx = -1$ if we approach the limit from below $x = 0$ (negative δx). We conclude that the derivative is not well-defined, as it depends on which direction you take the limit.

For a function $y(x)$ to be differentiable at a point x, the function must be both **continuous** and **smooth**.

4.1.2 Higher order derivatives

The derivative dy/dx is a function of x, so we can differentiate it again (assuming it is smooth and continuous). This is the *second derivative*, which is denoted by

$$\frac{d^2y}{dx^2} \equiv \frac{d}{dx}\left(\frac{dy}{dx}\right) , \tag{155}$$

It measures the rate of change of the slope, i.e. its *curvature*.

The notation for going further and taking the nth derivative is

$$\frac{d^ny}{dx^n} \equiv \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) . \tag{156}$$

So, for the example with $y = x^3$, we have

$$
\begin{aligned}
\frac{\mathrm{d}x^3}{\mathrm{d}x} &= 3x^2\,, \\
\frac{\mathrm{d}^2(x^3)}{\mathrm{d}x^2} &= \frac{\mathrm{d}(3x^2)}{\mathrm{d}x} = 6x\,, \\
\frac{\mathrm{d}^3(x^3)}{\mathrm{d}x^3} &= \frac{\mathrm{d}(6x)}{\mathrm{d}x} = 6\,, \\
\frac{\mathrm{d}^4(x^3)}{\mathrm{d}x^4} &= \frac{\mathrm{d}(6)}{\mathrm{d}x} = 0\,,
\end{aligned}
\tag{157}
$$

where all derivatives of higher order than the fourth are zero.

4.1.3 Alternative notations

The dy/dx notation for the derivative of $y(x)$ was proposed by Leibniz. However, Newton originally had a more compact notation using dots (or primes):

$$\dot{y} = \frac{dy}{dx} \quad \text{or} \quad y' = \frac{dy}{dx},$$
$$\ddot{y} = \frac{d^2y}{dx^2} \quad \text{or} \quad y'' = \frac{d^2y}{dx^2}.$$

For higher order derivatives it can be unwieldy to employ dots and dashes. Generally we use the more compact notation for the n^{th} derivative

$$y^{(n)}(x) = \frac{d^n y}{dx^n}.$$

Note that some people use Roman numerals with this convention, so that $d^4y/dx^4 = y^{\text{iv}}(x)$ and $d^5y/dx^5 = y^{\text{v}}(x)$.

4.2 Derivatives of elementary functions

Little progress is possible in calculus without knowing the basic derivatives of elementary functions, including powers of x, trigonometric, exponential and logarithmic functions.

You should have the following on automatic recall:

$$
\begin{aligned}
y &= x^n & \Rightarrow && \frac{dy}{dx} &= nx^{n-1}\,, \\
y &= e^x & \Rightarrow && \frac{dy}{dx} &= e^x, \\
y &= \ln x & \Rightarrow && \frac{dy}{dx} &= \frac{1}{x}\,, \\
y &= \sin x & \Rightarrow && \frac{dy}{dx} &= \cos x\,, \\
y &= \cos x & \Rightarrow && \frac{dy}{dx} &= -\sin x\,, \\
y &= \tan x & \Rightarrow && \frac{dy}{dx} &= \frac{1}{\cos^2 x}\,.
\end{aligned}
$$

You may not be as familiar with the derivatives of the hyperbolic functions introduced in Section 3:

$$y = \sinh x \qquad \Rightarrow \qquad \frac{dy}{dx} = \cosh x\,,$$
$$y = \cosh x \qquad \Rightarrow \qquad \frac{dy}{dx} = \sinh x,$$
$$y = \tanh x \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{1}{\cosh^2 x}\,.$$

However, these are easy to derive using the definitions of the hyperbolic functions. For example, differentiating $\sinh x$ we get

$$\frac{d\sinh x}{dx} = \frac{d}{dx}\left[\frac{1}{2}\left(e^{x} - e^{-x}\right)\right] = \frac{1}{2}\left(e^{x} + e^{-x}\right) = \cosh x\,.$$

The hyperbolic and trigonometric cases are similar, but note the sign difference between the derivatives of $\cosh x$ and $\cos x$.

4.3 Rules for differentiation

4.3.1 The product rule

Sometimes we are given a product of functions in the form

$$y(x) = u(x)v(x) \ , \tag{158}$$

where we know how to differentiate the factors $u(x)$ and $v(x)$ individually.

The rule for differentiating this product of functions is the following:

$$\boxed{\frac{\mathrm{d}(uv)}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}v + u\frac{\mathrm{d}v}{\mathrm{d}x}} \tag{159}$$

This result can be produced from first principles relatively quickly:

$$\begin{aligned}\frac{y(x+\delta x)-y(x)}{\delta x} &= \frac{u(x+\delta x)v(x+\delta x)-u(x)v(x)}{\delta x}\\ &= \frac{u(x+\delta x)v(x+\delta x)-u(x)v(x+\delta x)}{\delta x}\\ &\qquad +\frac{u(x)v(x+\delta x)-u(x)v(x)}{\delta x}\\ &= \left[\frac{u(x+\delta x)-u(x)}{\delta x}\right]v(x+\delta x)+u(x)\left[\frac{v(x+\delta x)-v(x)}{\delta x}\right].\end{aligned}$$

We now take the limit $\delta x \to 0$ and get the result.

<u>Example 4.1</u> Differentiate $y = \ln x \sin x$.

First set $u = \ln x$ and $v = \sin x$. Then:

$$\frac{d(\ln x \sin x)}{dx} = \frac{d \ln x}{dx} \sin x + \ln x \frac{d \sin x}{dx} = \frac{\sin x}{x} + \ln x \cos x.$$

4.3.2 The chain rule

Often we are given complicated expressions in which we have a function $y = f(u)$ with $u = u(x)$ itself being a function of x (e.g. $y = f(u) = \sin u$ and $u(x) = x^2$, so that $y = \sin x^2$). The method for differentiating a 'function of a function' is called the *chain rule* and is given by

$$\boxed{\frac{\mathrm{d}(f(u(x))}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u}\,\frac{\mathrm{d}u}{\mathrm{d}x}} \tag{160}$$

We can understand why this rule arises by writing

$$\frac{f(u(x+\delta x)) - f(u(x))}{\delta x} = \left[\frac{f(u(x+\delta x)) - f(u(x))}{u(x+\delta x) - u(x)}\right]\left[\frac{u(x+\delta x) - u(x)}{\delta x}\right]. \tag{161}$$

Next write $\delta u = u(x+\delta x) - u(x)$, i.e. the accompanying small change in the function u due to the small change δx in x.

We then have:

$$\frac{f(u(x+\delta x))-f(u(x))}{\delta x}=\left[\frac{f(u+\delta u)-f(u)}{\delta u}\right]\left[\frac{u(x+\delta x)-u(x)}{\delta x}\right], \tag{162}$$

and we now take the limit $\delta x \to 0$ (so that necessarily $\delta u \to 0$ as well). The first factor becomes df/du and the second factor du/dx.

Example 4.2 Differentiate $\sin x^2$ and $\ln(\cos x)$ with respect to x.

So we can write $f(u) = \sin u$ where $u = x^2$, and we want to find df/dx. First off, we have

$$\frac{df}{du} = \cos u, \qquad \frac{du}{dx} = 2x.$$

The using the chain rule we have:

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = (\cos x^2)2x.$$

Okay, for the second problem, let us set now $f(u) = \ln u$ and $u = \cos x$. We note:

$$\frac{df}{du} = \frac{1}{u}, \qquad \frac{du}{dx} = -\sin x,$$

thus

$$\frac{df}{dx} = \frac{1}{\cos x}(-\sin x) = -\tan x.$$

4.3.3 The quotient rule

We have already seen how to differentiate the product uv, now we consider the quotient u/v. In this case, we can find the derivative from the formula:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v(\mathrm{d}u/\mathrm{d}x) - u(\mathrm{d}v/\mathrm{d}x)}{v^2}} \tag{163}$$

This result comes about via the product and chain rules. First, write the quotient u/v as the product

$$\frac{u}{v} = u \times \left(\frac{1}{v}\right) . \tag{164}$$

Now differentiate (164) using the product rule (159) to give

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{\mathrm{d}u}{\mathrm{d}x} \times \left(\frac{1}{v}\right) + u\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{v}\right) . \tag{165}$$

Next use the chain rule (160) to calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{v}\right) = \frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{1}{v}\right)\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}x} . \tag{166}$$

Finally substitute this result back into (165) to find

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{1}{v}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{u}{v^2}\frac{\mathrm{d}v}{\mathrm{d}x} , \tag{167}$$

and group terms over the common denominator v^2.

Example 4.3 Differentiate $(\sin x)/x$ with respect to x.

$$\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{xd(\sin x)/dx - \sin x(dx/dx)}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

4.3.4 Implicit differentiation

It is also possible to find the derivative of y with respect to x from an equation of the form

$$g(y) = f(x)\,, \tag{168}$$

where g and f are given functions. For example: $\mathrm{e}^y \cos y = x \cos x$. Here, the exact dependence of y on x may not be known explicitly at all. Rather, it is *implicit*.

Using the chain rule:

$$\frac{\mathrm{d}g(y)}{\mathrm{d}x} = \frac{\mathrm{d}g(y)}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}\ , \tag{169}$$

so that differentiating (168) with respect to x we have

$$\frac{\mathrm{d}g(y)}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x}\ . \tag{170}$$

Rearranging, we find that

$$\boxed{g(y) = f(x) \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f/\mathrm{d}x}{\mathrm{d}g/\mathrm{d}y}\ .} \tag{171}$$

An important special case gives the 'reciprocal rule'. Suppose we want to differentiate $y(x)$ but only know the derivative of its inverse function $x(y)$, i.e. dx/dy.

Okay, set $f(x) = x$ in (168), so now $x = g(y)$. Then we have immediately

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y}}. \tag{172}$$

Example 4.4 Find the derivative of $y = \tan^{-1} x$ with respect to x.

Let us write $x = \tan y$ and then take the y derivative:

$$\frac{dx}{dy} = \frac{d\tan y}{dy} = \sec^2 y = 1 + \tan^2 y = 1 + x^2.$$

But the reciprocal rule says:

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{1 + x^2},$$

So $d\tan^{-1} x/dx = 1/(1 + x^2)$.

Example 4.5 A circle has equation $y^2 = 9 - (x - 1)^2$. Find the gradient.

We want to find an expression for dy/dx. The equation for a circle is of the form $g(y) = f(x)$, with $g = y^2$ and $f = 9 - (x - 1)^2$. The implicit differentiation law gives us

$$\frac{dy}{dx} = \frac{df/dx}{dg/dy} = \frac{-2(x-1)}{2y} = \pm\frac{(x-1)}{\sqrt{9-(x-1)^2}}.$$

Example 4.6 [2006 paper 2, Question 1A]. If

$$y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$

find dy/dx as a function of x.

First write $\sin y = x/\sqrt{1+x^2}$, which is in the form $g(y) = f(x)$ and use the implicit differentiation formula. We have $dg/dy = \cos y$. We also have, using the quotient rule:

$$\frac{df}{dx} = \frac{d}{dx}\frac{x}{\sqrt{1+x^2}} = \frac{\sqrt{1+x^2}(dx/dx) - (d\sqrt{1+x^2}/dx)x}{1+x^2},$$

$$= \frac{\sqrt{1+x^2} - [x(1+x^2)^{-1/2}]x}{1+x^2},$$

$$= \frac{1}{(1+x^2)^{3/2}}.$$

Putting the two results together:

$$\frac{dy}{dx} = \frac{df/dx}{dg/dy} = \frac{1}{\cos y(1+x^2)^{3/2}}.$$

It would be nice to have the RHS in terms of x only. To do this, note

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1 - \frac{x^2}{1+x^2}} = \frac{1}{\sqrt{1+x^2}}.$$

If we put this into our formula for dy/dx, we get simply:

$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

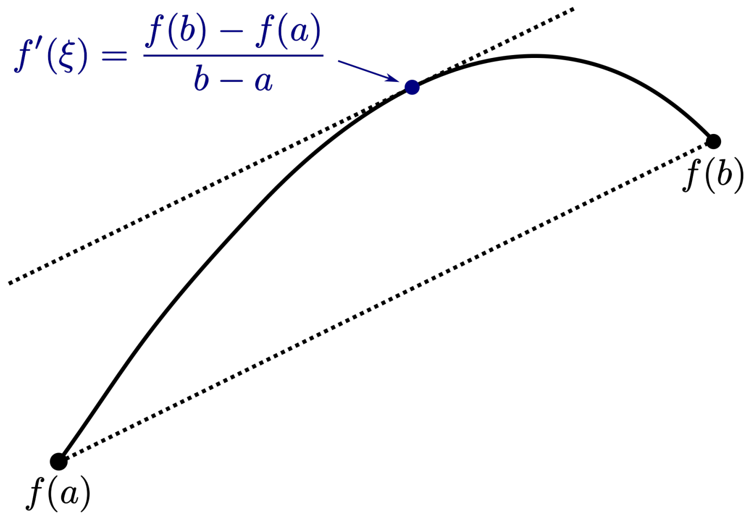


Figure 43: Example curve demonstrating the mean value theorem

4.4 The mean value theorem

One of the most useful, and intuitive, of the major results in real calculus is the mean value theorem. Consider a continuous curve given by $y = f(x)$ defined on the interval $a \le x \le b$. The theorem states that there exists at least one point $x = \xi$ in this interval so that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}. \tag{173}$$

In other words, at some point ξ the curve has the same slope as the line joining the two endpoints of the curve.

4.5 Stationary points

A stationary point (also called a 'turning point') of the curve $y = f(x)$ is a point where $\mathrm{d}y/\mathrm{d}x = 0$.

Stationary points can be classified using the following rules:

- If $\mathrm{d}^2y/\mathrm{d}x^2 > 0$ at the stationary point, then it is a **minimum**.

  This is because at a minimum the gradient *increases* through the turning point.

- If $\mathrm{d}^2y/\mathrm{d}x^2 < 0$ at the stationary point, then it is a **maximum**.

  This is because at a maximum the gradient *decreases* through the turning point.

- If $\mathrm{d}^2y/\mathrm{d}x^2 = 0$ at the stationary point, then further investigation is required:

1. if the first nonzero derivative $\mathrm{d}^ny/\mathrm{d}x^n \neq 0$ has n odd, then the stationary point is a **point of inflection**;
2. If the first nonzero derivative $\mathrm{d}^ny/\mathrm{d}x^n \neq 0$ has n even and it is positive, then the stationary point is a **minimum**;

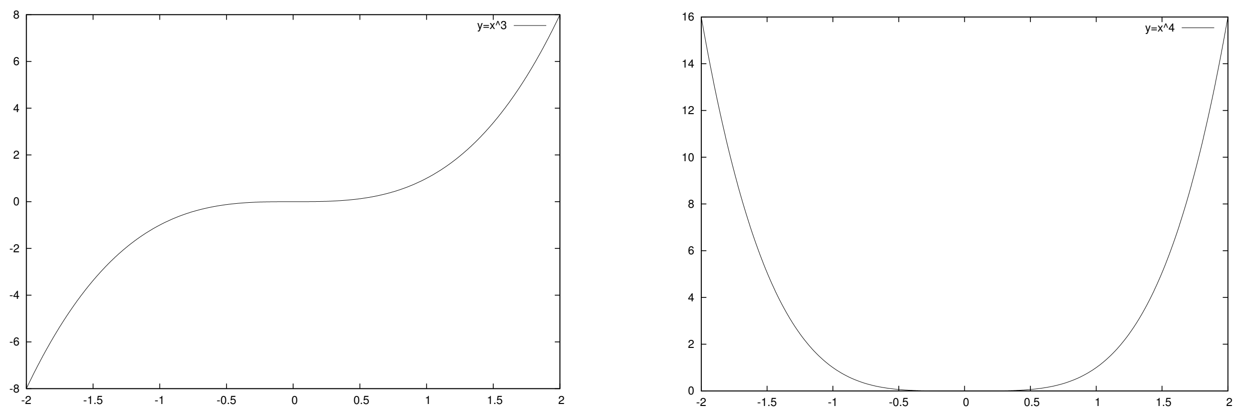


Figure 44: Graphs of $y = x^3$ and $y = x^4$.

3. If the first nonzero derivative $\mathrm{d}^n y/\mathrm{d}x^n \neq 0$ has n even and it is negative, then the stationary point is a **maximum**.

As examples, consider $y = x^3$ and $y = x^4$. Both have a stationary point at $x = 0$ with

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0\,.$$

For $y = x^3$, the point $x = 0$ is a point of inflection because

$$\frac{\mathrm{d}^3(x^3)}{\mathrm{d}x^3} = 6 \neq 0 \,. \tag{174}$$

For $y = x^4$, this is a minimum because the first non-zero derivative is even and positive,

$$\frac{\mathrm{d}^4 x^4}{\mathrm{d}x^4} = 24 > 0 \,. \tag{175}$$

4.5.1 More on points of inflection

An inflection point is where $d^2y/dx^2 = 0$ and also d^2y/dx^2 changes sign.

- It is where the curve changes from being concave up to concave down or vice versa.
- It need not be a stationary point (i.e. where $dy/dx = 0$).

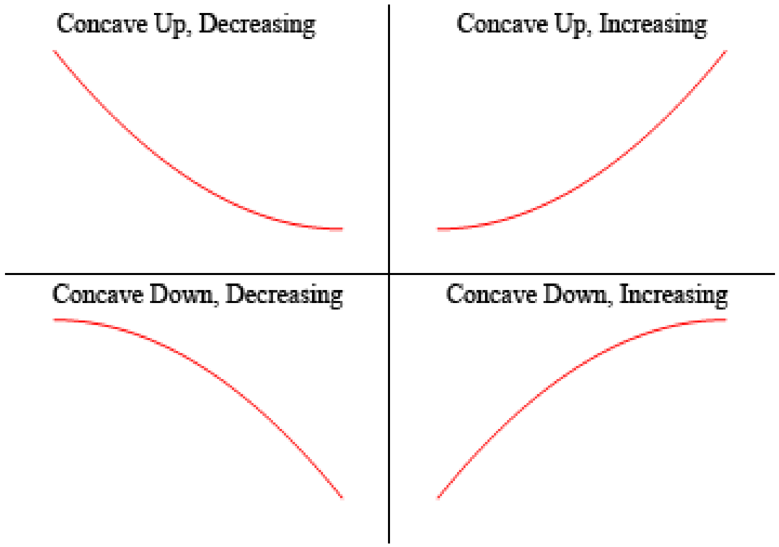


Figure 45: Concave up and concave down functions

- If $dy/dx > 0$ at the inflection point it is a 'rising point of inflection'; if $dy/dx < 0$ it is a 'falling point of inflection'.
- Between an adjacent maximum and a minimum there must be a point of inflection.

4.6 Curve sketching

Basic curve sketching techniques are very useful for detemining the *main features* of the overall shape of a function $y = f(x)$. It means we can understand the behaviour of the function without the need to compute it everywhere. In other words, we can get a qualitative idea of what it is about.

When sketching curves there are a number of things to consider:

1. Where does the curve **intercept** the x and y axes – i.e. what is the value of $f(0)$ and what are solutions for $f(x) = 0$?

2. Is there any **symmetry**? Is the function *even*, $f(x) = f(-x)$, or is the function *odd*, $f(x) = -f(-x)$?

3. What are the **asymptotes**? In other words, what is the behaviour as $x \to \pm\infty$ or at any boundaries?

4. Are there any **singularities**, that is, points where the function becomes infinite? These create *vertical asymptotes* about the singular point.

5. What are the **stationary points** (i.e. where does $dy/dx = 0$)? What is their nature – minimum, maximum or point of inflection?

The following examples illustrate important aspects of curve sketching techniques.

Example 4.7 Sketch $y = \exp x - \sin x$.

- Intercepts of the x axis are difficult to explicitly calculate: we need to solve $\exp x = \sin x$. However, graphically we can see that there are an infinite number of solutions, but only for $x < 0$.

  On the other hand, the curve cuts the y axis at $y = \exp(0) - \sin(0) = 1$.

- What about when $x \to \pm\infty$? Algebraically, we can see that as $x \to -\infty$

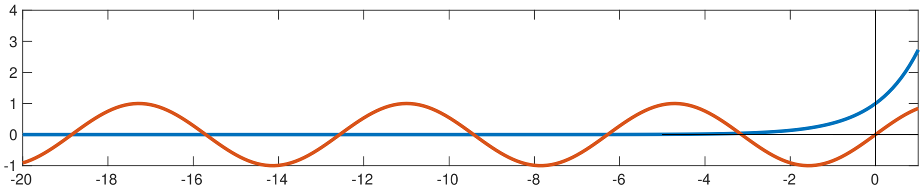


Figure 46: Graphs of $y = e^x$ and $y = \sin x$. The points at which they intercept give us the locations where $y = e^x - \sin x$ crosses the x axis.

then y approaches $-\sin x$. But when $x \to \infty$, we see that $y \to e^x$.

- What about stationary points? We see that $dy/dx = e^x - \cos x = 0$ also yields an infinite number of solutions with $x \leq 0$. Graphically we see that there is a turning point at $x = 0$. Is the $x = 0$ turning point a max or min? Well, we have $d^2y/dx^2 = e^x + \sin x$ which is $= 1$ at $x = 0$, and so this turning point is a minimum.

- We probably have enough information to sketch out the curve. Joining up $-\sin x$ and e^x with a minimum at $x = 0$.

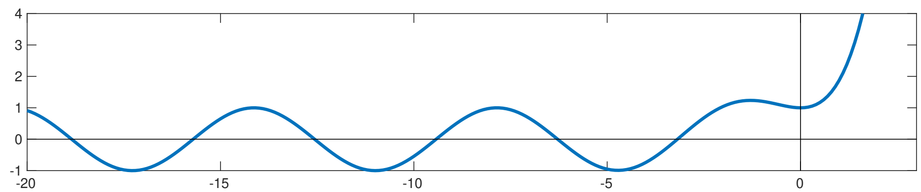


Figure 47: Graph of $y = e^x - \sin x$.

Example 4.8 Sketch $y = \frac{\ln x}{x}$. Show that $e^\pi > \pi^e$.

- The x axis intercept can be gleaned from $y = \ln x / x = 0$, and there is only one: $x = 1$. The curve cannot cross the y axis because of the $\ln x$ and the x in the denominator. And, in fact, the function is not even defined when $x < 0$.

- Asymptotic behaviour? Clearly as $x \to 0$, $y \to -\infty$. And when $x \to \infty$ the denominator in y defeats the numerator and $y \to 0$ from above.

- What about stationary points? Differentiating, we have

$$\frac{dy}{dx} = \frac{x(d\ln x/dx) - \ln x(dx/dx)}{x^2} = \frac{1-\ln x}{x^2}.$$

We have turning points wherever $dy/dx = 0$, i.e. when $\ln x = 1$ which yields $x = e$, with $y = 1/e$. So just one turning point.

Is it a max or min? We could look at the second derivative, but that would create more algebra. We do know that $y \to -\infty$ as $x \to 0$, and we know that $y \to 0$ as $x \to \infty$ from above, therefore the turning point inbetween these limits must be a *maximum*.

- We probably have enough to sketch out the curve now:

Let us derive this interesting inequality. We know that $y(x) < y(e)$ for all $x > 0$. So let us consider the point $x = \pi$. We then have $y(\pi) < y(e)$, which becomes

$$\frac{\ln \pi}{\pi} < \frac{\ln e}{e} = \frac{1}{e}.$$

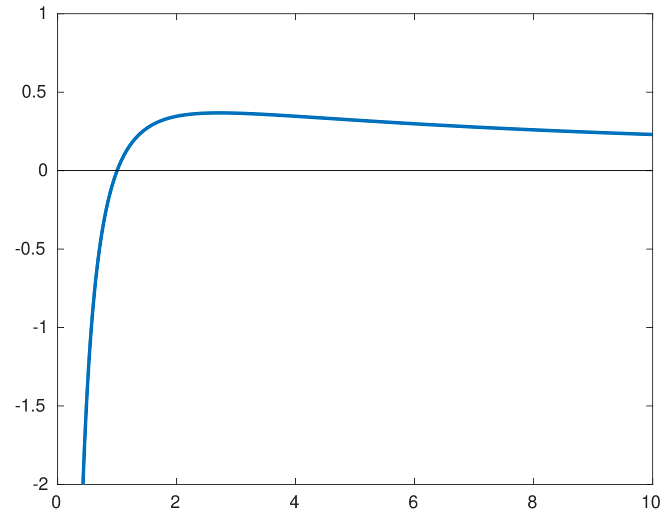


Figure 48: Graph of $y = \ln x / x$.

We rearrange this into $e \ln \pi < \pi$ and then $\ln \pi^e < \pi$. Take the exponential of both sides and we get

$$\pi^e < e^\pi,$$

as desired.

5 Integration

Integration is the *reverse* of differentiation; hence the study of calculus necessitates understanding one and the other.

But quite separately, integration is essential for a myriad of physical processes, involving areas, volumes, averaging, and correlations.

Integral equations (as opposed to differential equations) underpin applications as diverse as population dynamics (Volterra's equation), propagation of fish in a lake, viscoelasticity, electro and magneto-statics, and wave-scattering.

We can think of integration in two ways:

First, we have the *definite integral* over some interval bounded by the points $x = a$ and b:

$$\int_a^b f(x)dx \,. \tag{176}$$

This can be understood as the *area* (a) under the curve $y = f(x)$ and above the x-axis in the xy-plane and (b) bounded between the vertical lines $x = a$ and $x = b$.

Second, we have the *indefinite integral* which is expressed without bounds as

$$\int f(x)dx\,. \tag{177}$$

This can be regarded as the *inverse operation to differentiation* or the 'antiderivative', and it yields another function $F(x)$.

5.1 Integration as area

Consider the curve $y = f(x)$ in the range $a \leqslant x \leqslant b$. We can approximate the area under the curve by dividing the range up into N small subintervals of

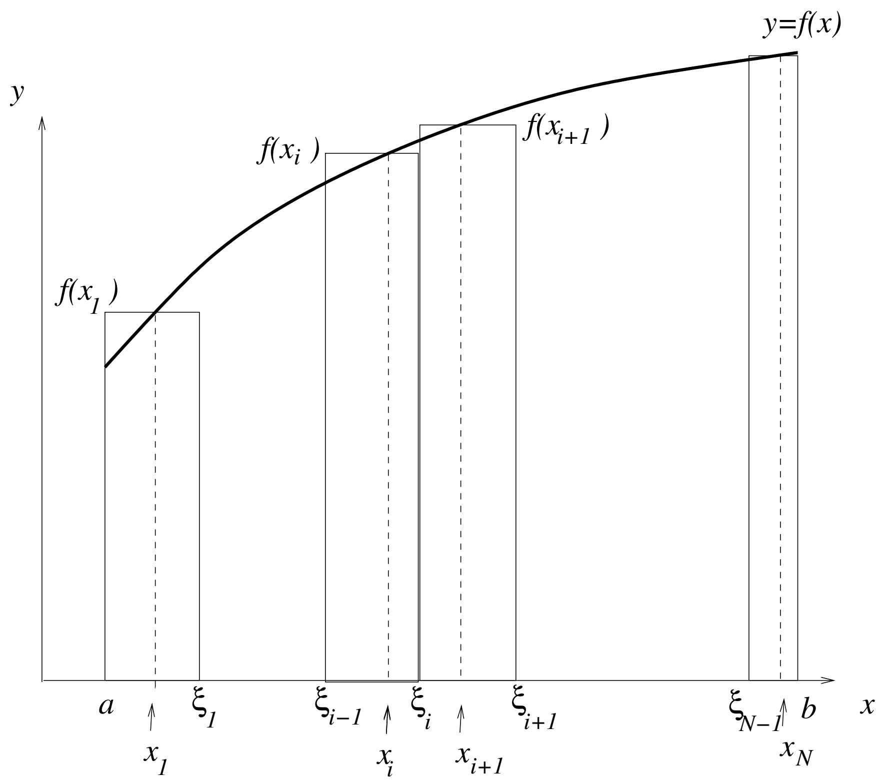


Figure 49: Approximation of the definite integral $\int_a^b f(x)\,\mathrm{d}x$.

length δx, with

$$\delta x = \frac{b - a}{N} \,, \tag{178}$$

so that the end points of the intervals are $\xi_0, \xi_1, .., \xi_N$ with

$$\begin{aligned} \xi_0 &= a \\ \xi_1 &= a + \delta x \\ \vdots & \quad \vdots \\ \xi_i &= a + i\delta x \\ \vdots & \quad \vdots \\ \xi_N &= b \,. \end{aligned} \tag{179}$$

We then choose N points $x_1, x_2, ..., x_N$, one lying in each of the subintervals, so that

$$\xi_{i-1} < x_i < \xi_i \,. \tag{180}$$

We next construct a rectangle on each subinterval, of length δx and height $f(x_i)$. The area of one rectangle is $\delta x f(x_i)$, and hence the total area of all the rectangles is

$$S = \sum_{i=1}^{N} \delta x f(x_i) \ . \tag{181}$$

The idea now is to take the limit $\delta x \to 0$, so that the length of each subinterval goes to zero while the number of subintervals goes to infinity ($N = (b-a)/\delta x \to \infty$).

The integral (if it exists) is then defined as

$$\boxed{\int_a^b f(x)\ \mathrm{d}x \equiv \lim_{\delta x \to 0} S\ ,} \tag{182}$$

and it corresponds to the area between the curve $y = f(x)$ and the x axis in the range $a \leqslant x \leqslant b$.

5.1.1 Integrability

For a given function $f(x)$, the integral (182) may not be well-defined on $a \leqslant x \leqslant b$. Determining whether $f(x)$ is *integrable* can be a complicated issue, but if the function is continuous and bounded on the finite interval then we can be sure the integral converges. Note that if $y = f(x)$ is singular in $a \leqslant x \leqslant b$, then the definite integral may or may not exist.

Integrals can also have an infinite range (e.g. $b \to \infty$). Provided the integrand converges rapidly enough then the integral can be well-defined.

5.1.2 Properties of integrals

Here are some important rules that follow from the definition (182)

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \qquad \text{(interval addition)}$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx, \qquad \text{(reverse/backward integration)}$$

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx, \quad \text{if } f(x) \le g(x), \ \text{(estimation)}.$$

5.2 Integration as the inverse of differentiation

5.2.1 The fundamental theorem of calculus

Consider the function $F(x)$ represented by the integral on an interval from a up to a variable x defined as

$$F(x) = \int_a^x f(u)\,\mathrm{d}u\ . \tag{183}$$

We can differentiate $F(x)$ from first principles using the procedure set out in section 4.1:

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}x} &= \lim_{\delta x \to 0} \frac{F(x+\delta x) - F(x)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{\int_a^{x+\delta x} f(u)\,\mathrm{d}u - \int_a^x f(u)\,\mathrm{d}u}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{\int_x^{x+\delta x} f(u)\,\mathrm{d}u}{\delta x}\ . \end{aligned} \tag{184}$$

Now as $\delta x \to 0$, the range of integration in the numerator gets shorter and shorter and the integrand can be approximated as a constant over that range. In fact, we have

$$\int_x^{x+\delta x} f(u)\,\mathrm{d}u \to \delta x f(x) \quad \text{as} \quad \delta x \to 0\ . \tag{185}$$

Substituting (185) into (184) yields

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{f(x)\delta x}{\delta x} = f(x)\ , \tag{186}$$

and we arrive at the (first) **fundamental theorem of calculus**:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(u)\,\mathrm{d}u = f(x)\ ,} \tag{187}$$

Thus differentiation undoes integration.

5.2.2 The second fundamental theorem of calculus

What about the reverse? This is covered by the *second fundamental theorem of calculus*: if we have a function $F(x)$ with derivative $F'(x)$, then

$$\boxed{\int_a^b F'(x)dx = F(b) - F(a)\,.} \tag{188}$$

The function $F'(x)$ is assumed to be integrable on the domain $[a, b]$. Thus integration can also undo differentiation.

A sketch of the proof goes something like this. We use equation (182) to rewrite the left side of (188) as a sum of rectangular areas as earlier,

$$\int_a^b F'(x)dx = \lim_{\delta x \to 0} \sum_{i=1}^{N} \delta x F'(x_i), \tag{189}$$

where we are free to choose the x_i that lie in each of the little subintervals, thus $\xi_{i-1} < x_i < \xi_i$, and recall that $\xi_i - \xi_{i-1} = \delta x$. But on each subinterval the *mean value theorem*, equation (173), states that there exists a point c_i so that $\xi_{i-1} < c_i < \xi_i$ and

$$F'(c_i) = \frac{F(\xi_i) - F(\xi_{i-1})}{\xi_i - \xi_{i-1}}.$$

Let us set $x_i = c_i$ for each i. We then get

$$\int_a^b F'(x)dx = \lim_{\delta x \to 0} \sum_{i=1}^{N} \delta x\, F'(c_i) = \lim_{\delta x \to 0} \sum_{i=1}^{N} [F(\xi_i) - F(\xi_{i-1})] = F(b) - F(a),$$

where all but two of the terms in the last sum cancel out.

5.2.3 Indefinite integrals

The integrals we have presented so far are called *definite integrals*, because they have definite limits (e.g., in equation 182 the lower limit is a and the upper limit b). Note that the value of the lower limit a in the first fundamental theorem of calculus (187) has no bearing on the result, so for instance

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{2a}^{x} f(u)\,\mathrm{d}u = f(x) \qquad (190)$$

as well.

It follows that there are an infinite number of different functions $F(x) = \int^{x} f(x)dx$ that we can differentiate with respect to x to give $f(x)$, and they all differ from each other only by an arbitrary additive constant. They are called *antiderivatives*.

The family of antiderivatives of $f(x)$ can be represented by an integral without specific limits $\int f(x)\,dx$, called the *indefinite integral*. The upper limit is understood to be x, and the lower limit is suppressed as it is understood to contribute to the arbitrary additive constant mentioned above.

5.3 Methods of integration

We will now review a whole series of common tricks and methods for evaluating indefinite integrals.

5.3.1 Reversal of differentiation

This is the simplest case, in which the integral can be done by inspection through prior knowledge of the appropriate derivative. For example, for the elementary functions described earlier we have the following:

$$\boxed{\begin{array}{ll} \dfrac{\mathrm{d}x^n}{\mathrm{d}x} = nx^{n-1} & \Rightarrow \int x^m \,\mathrm{d}x = \frac{x^{m+1}}{m+1} + c \text{ (if } \quad m \neq -1) \\ \\ \dfrac{\mathrm{d}\ln x}{\mathrm{d}x} = \frac{1}{x} & \Rightarrow \int \frac{1}{x}\,\mathrm{d}x = \ln x + c \\ \\ \dfrac{\mathrm{d}\exp(mx)}{\mathrm{d}x} = m\exp(mx) & \Rightarrow \int \exp(mx)\,\mathrm{d}x = \frac{\exp(mx)}{m} + c\ , \end{array}} \tag{191}$$

while for trigonometric and hyperbolic functions we have

$$\boxed{\begin{aligned}
\frac{\mathrm{d}\sin x}{\mathrm{d}x} &= \cos x && \Longrightarrow \int \cos x \;\mathrm{d}x = \sin x + c \\
\frac{\mathrm{d}\cos x}{\mathrm{d}x} &= -\sin x && \Longrightarrow \int \sin x \;\mathrm{d}x = -\cos x + c \\
\frac{\mathrm{d}\tan x}{\mathrm{d}x} &= \sec^2 x && \Longrightarrow \int \sec^2 x \;\mathrm{d}x = \tan x + c \\
\frac{\mathrm{d}\sinh x}{\mathrm{d}x} &= \cosh x && \Longrightarrow \int \cosh x \;\mathrm{d}x = \sinh x + c \\
\frac{\mathrm{d}\cosh x}{\mathrm{d}x} &= \sinh x && \Longrightarrow \int \sinh x \;\mathrm{d}x = \cosh x + c \\
\frac{\mathrm{d}\tanh x}{\mathrm{d}x} &= \mathrm{sech}^2 x && \Longrightarrow \int \mathrm{sech}^2 x \;\mathrm{d}x = \tanh x + c \;.
\end{aligned}} \tag{192}$$

5.3.2 Inverse trigonometric and hyperbolic functions

To differentiate $y = \sinh^{-1}(x/a)$ we first write $\sinh y = x/a$. Next we differentiate with respect to x

$$\begin{aligned} \frac{\mathrm{d}(\sinh y)}{\mathrm{d}x} &= \frac{1}{a} \\ \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x}\cosh y &= \frac{1}{a} \\ \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{a\cosh y} \\ &= \frac{1}{a\sqrt{\sinh^2 y + 1}} \\ &= \frac{1}{\sqrt{a^2 + x^2}}\ . \end{aligned}$$

In a similar way we can show that

$$\frac{\mathrm{d}(\cosh^{-1}(x/a))}{\mathrm{d}x} = \frac{1}{\sqrt{x^2 - a^2}}\ . \tag{193}$$

These two results lead us to the standard integrals

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \sinh^{-1}(x/a) + c\,, \qquad (194)$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \cosh^{-1}(x/a) + c\,. \qquad (195)$$

The inverse hyperbolic integrals (194) and (195) should be contrasted with the corresponding results for the trigonometric functions, e.g.

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a) + c\;. \qquad (196)$$

Note that $\cos^{-1}(x/a) = \pi/2 - \sin^{-1}(x/a)$.

5.3.3 Integrands of form $[f(x)]^\alpha \, \mathrm{d}f/\mathrm{d}x$

The following integral can be completed directly

$$\int \frac{\mathrm{d}f}{\mathrm{d}x}[f(x)]^\alpha \, \mathrm{d}x = \frac{1}{\alpha+1}[f(x)]^{\alpha+1} + c, \tag{197}$$

when $\alpha \neq -1$. This is the inverse of the chain rule (160). We will give some examples:

<u>Example 5.1</u> Complete the following: (a) $\int \cos x \sin^3 x \, dx$, (b) $\int (\tanh^6 x) \mathrm{sech}^2 x \, dx$, (c) $\int x \exp(-x^2) dx$.

(a)

$$\int \cos x \sin^3 x dx = \int \frac{1}{4}\frac{d \sin^4 x}{dx} dx = \frac{1}{4}\sin^4 x + c.$$

(b) Remember that $d \tanh x / dx = \mathrm{sech}^2 x$. Now we can do the following:

$$\int \tanh^6 x \, \mathrm{sech}^2 x dx = \int \frac{1}{7}\frac{d \tanh^7 x}{dx} dx = \frac{1}{7}\tanh^7 x + c.$$

(c)

$$\int xe^{-x^2}dx = \int -\frac{1}{2}\frac{de^{-x^2}}{dx}dx = -\frac{1}{2}e^{-x^2} + c.$$

The result (197) does not work when $\alpha = -1$, but it can be replaced by

$$\int \frac{\mathrm{d}f}{\mathrm{d}x}\frac{1}{f(x)}\,\mathrm{d}x = \ln[f(x)] + c\;. \tag{198}$$

<u>Example 5.2</u> Complete the following: (a) $\int x/(x^2+1)$, (b) $\int \tanh x$.

$$(a)\int \frac{x}{x^2+1}dx = \int \frac{1}{2}\frac{d\ln(x^2+1)}{dx}dx = \frac{1}{2}\ln(x^2+1) + c.$$

$$(b) \qquad \int \tanh x dx = \int \frac{\sinh x}{\cosh x} dx = \int \frac{1}{\cosh x} \frac{d \cosh x}{dx} dx,$$

$$= \int \frac{d}{dx} \left[\ln(\cosh x) \right] dx,$$

$$= \ln(\cosh x) + c.$$

5.3.4 Powers of trigonometric functions

Trigonometric identities are often useful. For instance, we can use the identities

$$\begin{aligned} \cos 2x &= 1 - 2\sin^2 x \;\Rightarrow\; \sin^2 x = \tfrac{1}{2}[1 - \cos 2x] \\ \cos 2x &= 2\cos^2 x - 1 \;\Rightarrow\; \cos^2 x = \tfrac{1}{2}[1 + \cos 2x] \end{aligned} . \qquad (199)$$

Then we can evaluate

$$\begin{aligned}\int \sin^4 x \, \mathrm{d}x &= \int \frac{1}{4}(1 - \cos 2x)^2 \, \mathrm{d}x \\ &= \int \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x) \, \mathrm{d}x \\ &= \int \frac{1}{4}\left(1 - 2\cos 2x + \frac{1}{2}[\cos 4x + 1]\right), \qquad (200)\end{aligned}$$

where the last step has been accomplished using (199) but with x replaced by $2x$. Each term in the integrand of (200) is now of an elementary form, and can be evaluated to give

$$\begin{aligned}\int \sin^4 x \, \mathrm{d}x &= \frac{1}{4}\left(x - \frac{2}{2}\sin 2x + \frac{1}{2}\left[\frac{1}{4}\sin 4x + x\right]\right) + c \\ &= \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c\,. \qquad (201)\end{aligned}$$

Odd powers can often be handled using the method outlined in Section 5.3.3.

For example:

$$\int \sin^3 x \, \mathrm{d}x = \int \sin x \sin^2 x \, \mathrm{d}x = \int \sin x (1 - \cos^2 x) \, \mathrm{d}x = -\cos x + \frac{\cos^3 x}{3} + c \, . \tag{202}$$

<u>Example 5.3</u> Complete the following: (a) $\int \cos^4 x \, dx$, (b) $\int \tan^5 x \, dx$.

$$\begin{aligned}
\int \cos^4 x dx &= \frac{1}{4} \int (1 + \cos 2x)^2 dx, \\
&= \frac{1}{4} \int 1 + 2\cos 2x + \cos^2 2x \, dx, \\
&= \frac{1}{4} \int 1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x) \, dx, \\
&= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c.
\end{aligned}$$

$$\begin{aligned}\int \tan^5 x dx = \int \tan^3 x \, \tan^2 x \, dx &= \int \tan^3 x(\sec^2 x - 1)dx, \\ &= \int \tan^3 x \sec^2 x - \tan x \, \tan^2 x \, dx, \\ &= \frac{1}{4}\tan^4 x - \int \tan x(\sec^2 x - 1)dx, \\ &= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln(\cos x) + c\end{aligned}$$

5.3.5 Partial fractions

Consider the integral

$$\int \frac{1}{x^2 + x} \, \mathrm{d}x \, . \tag{203}$$

We can make progress if we split the integrand into its partial fractions, i.e.

$$\frac{1}{x^2 + x} = \frac{1}{x(x + 1)} = \frac{\alpha}{x} + \frac{\beta}{x + 1} \tag{204}$$

for constants α and β. To find α and β we write

$$\frac{\alpha}{x} + \frac{\beta}{x+1} = \frac{\alpha(x+1) + \beta x}{x(x+1)} = \frac{x(\alpha+\beta) + \alpha}{x(x+1)} \,, \tag{205}$$

and then comparing the numerator of the final expression in (205) with the integrand in (203) we see that $\alpha + \beta = 0$ and $\alpha = 1$ (implying $\beta = -1$), so that

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1} \,. \tag{206}$$

The integral (203) then becomes

$$\begin{aligned} \int \frac{1}{x^2+x} \, \mathrm{d}x &= \int \frac{1}{x} \, \mathrm{d}x - \int \frac{1}{x+1} \, \mathrm{d}x \\ &= \ln x - \ln(x+1) + c \,. \end{aligned} \tag{207}$$

<u>Example 5.4</u> Evaluate

$$\int \frac{x+1}{1 - x + x^2 - x^3} \, \mathrm{d}x \,. \tag{208}$$

We re-express the integrand in partial fractions, first factorising the denominator:

$$\begin{aligned}\frac{x+1}{1-x+x^2-x^3} = \frac{x+1}{(1-x)(1+x^2)} &= \frac{A}{1-x} + \frac{Bx+C}{1+x^2}, \\ &= \frac{A + Ax^2 + Bx + C - Bx^2 - Cx}{(1-x)(1+x^2)}, \\ &= \frac{A + C + (B-C)x + (A-B)x^2}{(1-x)(1+x^2)},\end{aligned}$$

where A, B, and C are constants to be determined. In the numerators of the left and right sides, the coefficients of 1, x, and x^2 can be equated giving the equations:

$$A + C = 1, \quad B - C = 1, \quad A - B = 0.$$

These can be solved quickly to give us $A = B = 1$ and $C = 0$. We are now in a good place to evaluate the integral:

$$\int \frac{x+1}{1-x+x^2+x^3}\,dx = \int -\frac{1}{x-1} + \frac{x}{1+x^2}\,dx = -\ln(x-1) + \tfrac{1}{2}\ln(1+x^2) + c.$$

5.3.6 Trigonometric and other substitutions

Difficult integrals can often be simplified by changing variables. The trick is to know which substitution to use! The main point is to bring the integral to a recognised form where the integral can be done by inspection using known results. This technique is best developed by doing lots of different examples.

Consider the exponential integral

$$\int xe^{-x^2}\,dx. \tag{209}$$

Think of this as

$$\int f(u(x))u'(x)\,dx = \int f(u)\,du = F(u) + c = F(u(x)) + c, \tag{210}$$

where F is the anti derivative of f. Setting $u = x^2$ and thus $du = 2x\,dx$ gives

$$\int xe^{-x^2}dx = \frac{1}{2}\int e^{-u}du = -\frac{1}{2}e^{-u} + \text{const.} = -\frac{1}{2}e^{-x^2} + \text{const.} \tag{211}$$

It is important in a definite integral to also change the limits to match the new variable u.

Commonly, one can employ substitutions of trigonometric functions. Consider

$$\int \frac{1}{1+x^2}\, \mathrm{d}x \ . \tag{212}$$

Try $x = \tan t$, so differentiating

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sec^2 t \ ,$$

which allows the replacement in (212)

$$\mathrm{d}x = \sec^2 t \ \mathrm{d}t \ . \tag{213}$$

In this way

$$\int \frac{1}{1+x^2}\, \mathrm{d}x = \int \frac{1}{1+\tan^2 t} \sec^2 \mathrm{t} \ \mathrm{d}t = \int 1 \ \mathrm{d}t \ , \tag{214}$$

where the last step has been accomplished simply via the identity $\tan^2 t + 1 = \sec^2 t$.

The integral on the right of (214) is now very easy indeed, and we find that

$$\boxed{\int \frac{1}{1+x^2}\, \mathrm{d}x = t + c = \tan^{-1} x + c\,.} \tag{215}$$

When encountering integrands with quadratics in the denominator, we can often *complete the square* in order to make the trigonometric substitutions above.

As an example, consider the following integral

$$\int \frac{dx}{\sqrt{3+2x-x^2}} = \int \frac{dx}{\sqrt{4-(x-1)^2}} = \int \frac{du}{\sqrt{4-u^2}}\,, \tag{216}$$

where we have substituted $u = x - 1$ and $du = dx$. Now substitute $u = 2\sin\, t$ with $du = 2\cos\, t\, dt$ to find

$$\int \frac{du}{\sqrt{4-u^2}} = \int \frac{2\cos t\, dt}{2\cos t} = t + \text{const.} = \sin^{-1}\left(\frac{x-1}{2}\right) + \text{const.} \tag{217}$$

Another useful trick when dealing with integrals of awkward trigonometric functions is to use the *half-angle formula*.

Start with the substitution

$$\tan(x/2) = t \ . \tag{218}$$

Then we can show that

$$\sin x = 2\sin(x/2)\cos(x/2) = \frac{2\tan(x/2)}{\sqrt{1+\tan^2(x/2)}\sqrt{1+\tan^2(x/2)}} = \frac{2t}{1+t^2} \ , \tag{219}$$

Similarly,

$$\cos x = 2\cos^2(x/2) - 1 = \frac{2}{\sec^2(x/2)} - 1 = \frac{2}{1+\tan^2(x/2)} - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2} \ , \tag{220}$$

and

$$\tan x = \frac{\sin x}{\cos x} = \frac{2t}{1-t^2}\ . \tag{221}$$

Finally we need a way to re-express dx in terms of dt. We have

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right) = \frac{1}{2}(1+t^2).$$

Thus

$$dx = \frac{2}{1+t^2}\,dt. \tag{222}$$

<u>Example 5.5</u> Use the substitution $t = \tan(x/2)$ to evaluate $\int \sec x \, \mathrm{d}x$.

$$\int \sec x dx = \int \frac{1+t^2}{1-t^2}\,\frac{2}{1+t^2}\,dt = \int \frac{2}{1-t^2}\,dt.$$

We now use partial fractions to re-express the integrand. Skipping a few steps, which you should supply,

$$\frac{2}{1-t^2} = \frac{2}{(1-t)(1+t)} = \frac{1}{1-t} + \frac{1}{1+t}.$$

Now we can evaluate the integral:

$$\int \sec x \, dx = \int \frac{1}{1-t} + \frac{1}{1+t} \, dt = -\ln(1-t) + \ln(1+t) + c.$$
$$= \ln\left(\frac{1+t}{1-t}\right) + c = \ln\left(\frac{1+\tan(x/2)}{1-\tan(x/2)}\right) + c.$$

5.3.7 Integration by parts

Integration by parts is closely related to the product rule, which we now rearrange and then integrate between the limits $a \leqslant x \leqslant b$:

$$\int_a^b u \frac{\mathrm{d}v}{\mathrm{d}x} \, \mathrm{d}x = \int_a^b \frac{\mathrm{d}(uv)}{\mathrm{d}x} \, \mathrm{d}x - \int_a^b \frac{\mathrm{d}u}{\mathrm{d}x} v \, \mathrm{d}x \, . \tag{223}$$

The first integral on the right hand side can be completed (using the fundamental theorem of calculus), and we are then left with the rule for **integration by**

parts

$$\boxed{\int_a^b u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \Big[uv\Big]_a^b - \int_a^b \frac{\mathrm{d}u}{\mathrm{d}x}v\,\mathrm{d}x\ .} \tag{224}$$

where

$$\Big[uv\Big]_a^b = u(b)v(b) - u(a)v(a).$$

Example 5.6 Evaluate $\int \ln x\,\mathrm{d}x$ and $\int x\sec^2 x\,\mathrm{d}x$.

$$\int \ln x dx = \int \frac{dx}{dx}\ln x dx = x\ln x - \int x\frac{d\ln x}{dx}\,dx,$$

$$= x\ln x - \int 1\,dx = x\ln x - x + c.$$

$$\int x\sec^2 x\,dx = \int x\frac{d\tan x}{dx}\,dx = x\tan x - \int \frac{dx}{dx}\tan x\,dx,$$

$$= x\tan x - \int \tan x dx = x\tan x + \ln(\cos x) + c.$$

5.3.8 Symmetry - integrating even and odd functions

When a function is described as being *even* or *odd* then this refers to its symmetry or antisymmetry about the y-axis. Specifically

$$\begin{aligned} f(x) &= f(-x) \quad \textbf{even function,} \\ f(x) &= -f(-x) \quad \textbf{odd function,} \end{aligned} \tag{225}$$

For example $f(x) = \cos x$ is an even function, $f(x) = \sin x$ is an odd function. Note that for an odd function $f(0) = -f(-0)$, so that $f(0) = 0$. Note also that most functions are neither odd nor even!

Evenness or oddness can be important when integrating.

For instance, without doing detailed calculations we can see straight away that

$$\int_{-\pi/4}^{\pi/4} \frac{x}{1+x^2}\,\mathrm{d}x = 0\ . \tag{226}$$

The reason for this is that the integrand in (226) is an **odd** function of x, so that the area under the curve for $x > 0$ exactly cancels out with the area under the curve for $x < 0$, to give a total area of zero.

Another example, this time involving infinite limits, is

$$\int_{-\infty}^{\infty} \frac{\sin x}{1 + x^2} \, \mathrm{d}x = 0 \; , \tag{227}$$

because again the integrand is an **odd** function of x.

In both cases this works because the integration interval itself is symmetric around $x = 0$. If we integrated between $x = 1$ and $x = 2$ (for example) then the integrals above would not necessarily be zero.

Of course, for **even** functions things are different because now the areas on either side of $x = 0$ add up rather than cancel. For example,

$$\int_{-1}^{1} \frac{x^2}{1 + x^2} \, \mathrm{d}x = 2 \int_{0}^{1} \frac{x^2}{1 + x^2} \, \mathrm{d}x \; , \tag{228}$$

while

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,\mathrm{d}x = 2\int_{0}^{\infty} \frac{\cos x}{1+x^2}\,\mathrm{d}x\ . \tag{229}$$

In (228) the integral on the right hand side can be calculated using the substi-

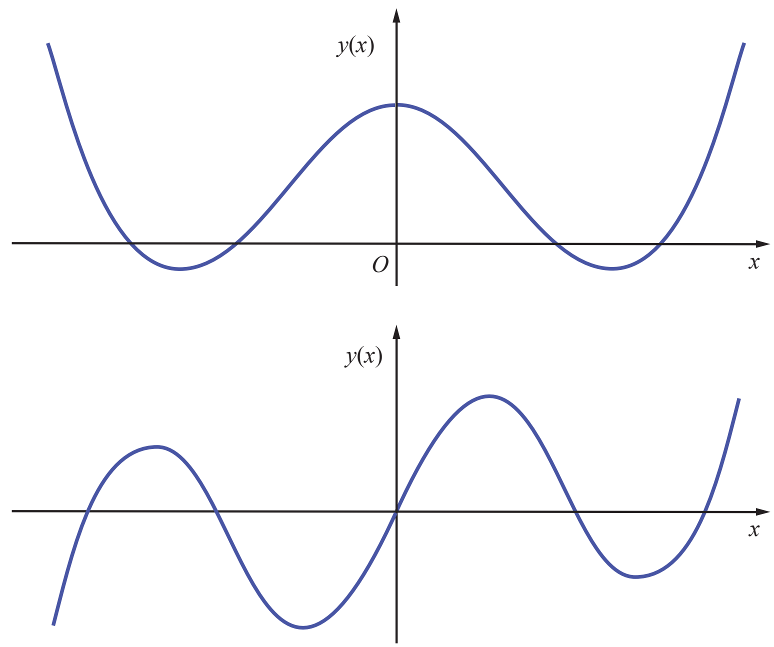


Figure 50: Even (top) and odd functions.

tution $x = \tan y$, to give

$$\begin{aligned} 2\int_0^1 \frac{x^2}{1+x^2}\,\mathrm{d}x &= 2\int_0^{\pi/4} \frac{\tan^2 y}{1+\tan^2 y}\sec^2 y\;\mathrm{d}y = 2\int_0^{\pi/4} \tan^2 y\;\mathrm{d}y \\ &= 2\int_0^{\pi/4} (\sec^2 y - 1)\;\mathrm{d}y \\ &= 2\Big[\tan y - y\Big]_0^{\pi/4} = 2 - \frac{\pi}{2}\,. \end{aligned} \tag{230}$$

Unfortunately, the integral on the right hand side in (229) requires more advanced methods beyond the scope of this course (cf. the Cauchy residue theorem).

5.3.9 Reduction formulae

Reduction formulae are often used to reduce a complicated integral down to something more manageable.

For instance, suppose we wanted to know $\int_0^{\pi/2} \sin^{10} x \, \mathrm{d}x$ or $\int_0^{\pi/2} \sin^{1000} x \, \mathrm{d}x$. Before tackling these individually, consider:

$$I_{2n} \equiv \int_0^{\pi/2} \sin^{2n} x \; \mathrm{d}x \tag{231}$$

for n a positive integer.

The idea is to relate I_{2n} to a similar integral with a lower power of $\sin x$. We write

$$I_{2n} = \int_0^{\pi/2} \sin x \sin^{2n-1} x \; \mathrm{d}x \; , \tag{232}$$

and then use integration by parts (with $u = \sin^{2n-1} x$ and $\mathrm{d}v/\mathrm{d}x = \sin x$) to get

$$\int_0^{\pi/2} \sin x \sin^{2n-1} x \, \mathrm{d}x = \Big[-\cos x \sin^{2n-1} x \Big]_0^{\pi/2} + (2n-1) \int_0^{\pi/2} \cos^2 x \sin^{2n-2} x \, \mathrm{d}x$$
$$= (2n-1) \int_0^{\pi/2} (1 - \sin^2 x) \sin^{2n-2} x \, \mathrm{d}x$$
$$= (2n-1) \int_0^{\pi/2} \sin^{2n-2} x \, \mathrm{d}x - (2n-1) \int_0^{\pi/2} \sin^{2n} x \, \mathrm{d}x \,.$$

In other words, we have

$$I_{2n} = (2n-1) I_{2n-2} - (2n-1) I_{2n} \,, \tag{233}$$

or rearranging

$$I_{2n} = \frac{(2n-1)}{2n} I_{2n-2} \,. \tag{234}$$

Equation (234) is a *recurrence relation*, which means we can just apply it over and over again, so

$$\begin{aligned} I_{2n} &= \frac{(2n-1)}{2n} I_{2n-2} \\ &= \frac{(2n-1)}{2n} . \frac{(2n-3)}{2n-2} I_{2n-4} \\ &= \frac{(2n-1)}{2n} . \frac{(2n-3)}{2n-2} . \frac{(2n-5)}{2n-4} I_{2n-6} \\ &= \ldots . \end{aligned} \tag{235}$$

Note how the index on the integral goes down by 2 each time. If we do this operation n times then the index of the integral goes down to zero and we end up with

$$I_{2n} = \frac{(2n-1)(2n-3)(2n-5)\ldots \times 3 \times 1}{(2n)(2n-2)(2n-4)\ldots \times 4 \times 2} I_0 \ . \tag{236}$$

The point now is that I_0 is very easy to calculate, because

$$I_0 = \int_0^{\pi/2} \sin^0 x \, \mathrm{d}x = \int_0^{\pi/2} 1 \, \mathrm{d}x = \pi/2 \ , \tag{237}$$

and putting (237) back into (236) gives us a closed expression for I_{2n}.

Therefore, back to our original query, we have $\int_0^{\pi/2} \sin^{10} x \, \mathrm{d}x = (63/512)\pi$.

(We also get $\int_0^{\pi/2} \sin^{1000} x \, \mathrm{d}x = 0.0396234\ldots$.)

<u>Example 5.7</u> [2005, paper 2, question 11F]

(a) By expressing the integrand in partial fractions, evaluate

$$\int_1^2 \frac{3x^2 + 5x + 1}{x(x+1)(x+2)} \, \mathrm{d}x \ .$$

(b) Evaluate the definite integral

$$\int_0^{\pi/4} \frac{1}{1 + \cos 2\theta} \, \mathrm{d}\theta \ .$$

(c) Using your results from (b), or otherwise, evaluate the definite integral

$$\int_0^{\pi/2} \frac{1}{1 + \sin \phi} \, \mathrm{d}\phi \ .$$

First let us rewrite the integrand in terms of several fractions:

$$\begin{aligned}\frac{3x^2+5x+1}{x(x+1)(x+2)} &= \frac{A}{x}+\frac{B}{x+1}+\frac{C}{x+2}, \\ &= \frac{A(x+1)(x+2)+Bx(x+2)+Cx(x+1)}{x(x+1)(x+2)}, \\ &= \frac{(A+B+C)x^2+(3A+2B+C)x+2A}{x(x+1)(x+2)}.\end{aligned}$$

Equating the coefficients of x^2, x, and 1 on the numerators of both sides of the equation gives us three equations for A, B, and C:

$$A+B+C=3, \quad 3A+2B+C=5, \quad 2A=1.$$

We see that $A=1/2$ straightaway. Subtracting the first two equations from each other gives $B=1$, and then $C=3/2$ follows. Now we go to the integral itself:

$$\int_1^2 \frac{3x^2+5x+1}{x(x+1)(x+2)}dx = \int_1^2 \frac{1/2}{x} + \frac{1}{x+1} + \frac{3/2}{x+2}\, dx,$$
$$= \left[\frac{1}{2}\ln x + \ln(x+1) + \frac{3}{2}\ln(x+2)\right]_1^2,$$
$$= \frac{1}{2}(\ln 2 - \ln 1) + \ln 3 - \ln 2 + \frac{3}{2}(\ln 4 - \ln 3),$$
$$= \ln\sqrt{\frac{32}{3}}.$$

(b) Recall that $\cos 2\theta = 2\cos^2\theta - 1$, so our integral becomes

$$\int_0^{\pi/4} \frac{d\theta}{1+\cos 2\theta} = \int_0^{\pi/4} \frac{d\theta}{2\cos^2\theta} = \frac{1}{2}\int_0^{\pi/4} \sec^2\theta\, d\theta,$$
$$= \frac{1}{2}\left[\tan\theta\right]_0^{\pi/4} = \frac{1}{2}\tan(\pi/4) = \frac{1}{2}.$$

(c) Obviously, we would like to transform in some way $\sin\phi$ into $\cos 2\theta$. Recall that $\sin(\pi/2 - x) = \cos x$, so how about the substitution:

$$\phi = \frac{\pi}{2} - 2\theta\,?$$

Then $d\phi/d\theta = -2$, which means $d\phi = -2d\theta$. The integration limits need changing too: from $[0,\, \pi/2]$ to $[\pi/4,\, 0]$. We then have

$$\int_0^{\pi/2} \frac{d\phi}{1 + \sin\phi} = \int_{\pi/4}^{0} \frac{-2d\theta}{1 + \sin(\pi/2 - 2\theta)} = 2\int_0^{\pi/4} \frac{d\theta}{1 + \cos 2\theta} = 1,$$

where the last equality comes about because of part (b).

6 Taylor Series

You are probably already familiar with infinite series. Power series (and the Taylor series we will be studying) are a special type of infinite series where each term is proportional to a power of a variable x.

- They can be used to approximate values of a function $f(x)$ near a point $x = a$ (say) using the derivatives of the function at that point, that is, $f'(a)$, $f''(a)$, $f'''(a)$ etc.

- Power series can also be used to approximate functions themselves or, more usually, solutions to differential equations. Their advantage in solving such equations issues from the ease with which power series can be differentiated and integrated, since term by term they are simply powers x^n.

- Power series are also ubiquitous throughout numerical analysis. Every time you press a function button on a calculator (e.g. to find $\sin(\pi/5)$), you are usually summing up a rapidly convergent power series. The series is truncated after a few terms when the result has sufficient accuracy.

6.1 Approximating a function

We want to approximate a function $f(x)$ near a point $x = a$ using its derivatives at that point. Specifically, we want to find $f(x+h)$, where h is a small distance from $x = a$.

A very crude estimate for $f(x+h)$ can be obtained from a linear approximation:

$$f(a+h) \approx f(a) + hf'(a) \ . \tag{238}$$

What we are doing here is approximating the function $f(x)$ near the point $x = a$ by the tangent line at that point: $f'(a) = (y - f(a))/(x - a)$. (See figure.)

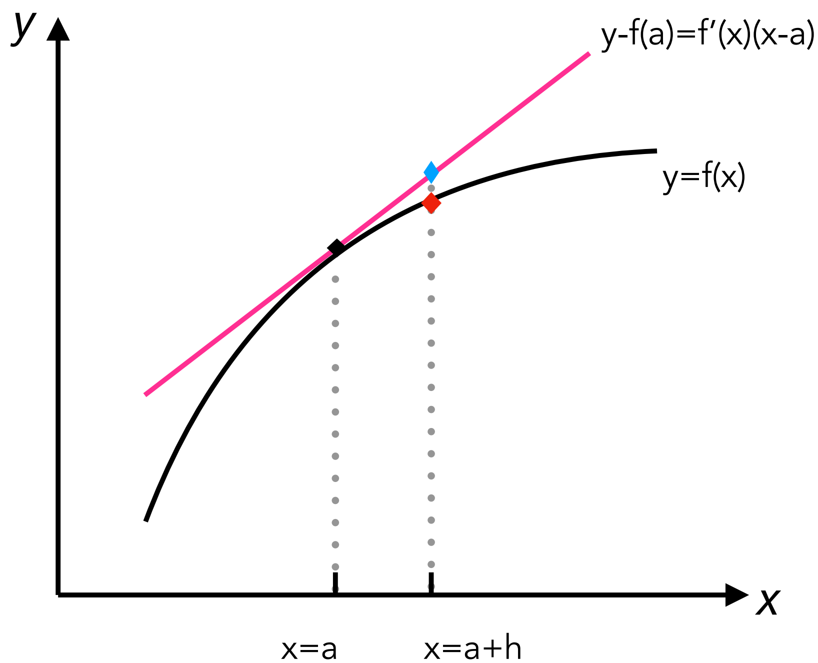


Figure 51: Linear approximation (blue diamond) to $f(a+h)$ (red diamond), using the tangent line at $x=a$.

Then we are evaluating the approximated function at $x = a + h$.

Of course, the smaller h is, then the better the approximation. But let's keep h fixed and try and improve the approximation.

We can rewrite (238) in a slightly different way, by setting $x = a + h$:

$$f(x) \approx f(a) + (x - a)f'(a) \; . \tag{239}$$

Now we are approximating the function itself $f(x)$ rather than its value at a specific point.

We can do the same with $f'(x)$:

$$f'(x) \approx f'(a) + (x - a)f''(a). \tag{240}$$

Now we employ the (second) Fundamental Theorem of Calculus,

$$\int_a^{a+h} f'(x)\,\mathrm{d}x = f(a + h) - f(a) \; . \tag{241}$$

Substitute the approximation (240) into the left side of (241), to give

$$\begin{aligned}\int_a^{a+h} f'(x)\,\mathrm{d}x &\approx \int_a^{a+h} [f'(a) + (x-a)f''(a)]\,\mathrm{d}x \\ &\approx \left[x f'(a) + \frac{(x-a)^2}{2} f''(a) \right]_a^{a+h} \\ &\approx h f'(a) + \frac{h^2}{2} f''(a)\ , \end{aligned} \tag{242}$$

and putting approximation (242) back into (241) gives

$$f(a+h) \approx f(a) + h f'(a) + \frac{h^2}{2} f''(a)\ . \tag{243}$$

Equation (243) is a *second-order* approximation for $f(a+h)$ (since it involves h squared), and it is an *improvement* over the first-order approximation (238). Now we are approximating the function $f(x)$ with a parabola.

We can derive higher-order approximations: if we use (243) to give a second-order approximation for $f'(x)$ in the form

$$f'(x) \approx f'(a) + (x-a)f''(a) + \frac{(x-a)^2}{2} f'''(a) \ , \tag{244}$$

and putting (244) into the left of (241) then gives

$$\begin{aligned} \int_a^{a+h} f'(x)\,\mathrm{d}x &\approx \int_a^{a+h} \left[f'(a) + (x-a)f''(a) + \frac{(x-a)^2}{2} f'''(a) \right] \mathrm{d}x \\ &= hf'(a) + \frac{h^2}{2} f''(a) + \frac{h^3}{6} f'''(a) \ . \end{aligned} \tag{245}$$

Now putting approximation (245) into (241) gets us

$$f(a+h) \approx f(a) + hf'(a) + \frac{h^2}{2} f''(a) + \frac{h^3}{6} f'''(a) \ . \tag{246}$$

Equation (246) is a *third-order* approximation for $f(a+h)$ (since it involves h to the third power) and it is, in turn, an improvement over the second-order approximation (243). We are approximating the function now with a cubic polynomial.

We can carry on doing this for as many times as $f(x)$ is differentiable at $x = a$, and find that the nth order approximation is

$$\boxed{\begin{aligned} f(a+h) \approx{}& f(a) + hf'(a) + \tfrac{h^2}{2}f''(a) + \tfrac{h^3}{6}f'''(a) + \tfrac{h^4}{24}f^{(4)}(a) \\ &+ \tfrac{h^5}{120}f^{(5)}(a) + \cdots + \tfrac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \tfrac{h^n}{n!}f^{(n)}(a)\ . \end{aligned}} \tag{247}$$

6.2 Taylor's theorem

Equation (247) is an approximate result but (hopefully) the error involved gets smaller as more and more term are included.

This is in fact guaranteed by **Taylor's Theorem**. We state the exact result

$$\begin{aligned} f(a+h) ={}& f(a) + hf'(a) + \frac{h^2}{2}f''(a) + \frac{h^3}{6}f'''(a) + \frac{h^4}{24}f^{(4)}(a) \\ &+ \frac{h^5}{120}f^{(5)}(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + R_{n+1}\ , \end{aligned} \tag{248}$$

where R_{n+1} is the **remainder term**. Taylor's Theorem states that, provided f

can be differentiated $n+1$ times, there exists some point $x=\zeta$ which lies in the range $a<\zeta<a+h$ such that

$$R_{n+1}=\frac{h^{n+1}}{(n+1)!}f^{(n+1)}(\zeta)\ . \tag{249}$$

What this means is that the error in approximating $f(a+h)$ by the nth order approximation (247) is R_{n+1}, and that the size of this error is proportional to h^{n+1}.

Equation (248) is a Taylor expansion about the point a. It is often written in the alternative, but completely equivalent, way as

$$\boxed{\begin{aligned} f(x) = \quad & f(a)+(x-a)f'(a)+\tfrac{(x-a)^2}{2}f''(a)+\tfrac{(x-a)^3}{6}f'''(a) \\ & +\tfrac{(x-a)^4}{24}f^{(4)}(a)+\tfrac{(x-a)^5}{120}f^{(5)}(a)+\ldots.+\tfrac{(x-a)^n}{n!}f^{(n)}(a)+R_{n+1} \end{aligned}} \tag{250}$$

Here we are rewriting the function itself as a sum, rather than its value at a specific point. This sum (minus the remainder) is called a *Taylor polynomial*. As an example, let's derive an approximation for $e^{1/2}$, an actual number.

Write $f(x) = \exp(x)$, $a = 0$ and $h = 1/2$. First note

$$f'(0) = f''(0) = f'''(0) = ... = f^{(n)}(0) = e^0 = 1 \,. \tag{251}$$

Plugging all this into (248) gives

$$\exp(1/2) = 1 + \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{6}\left(\frac{1}{2}\right)^3 + \frac{1}{24}\left(\frac{1}{2}\right)^4 + ... + \frac{1}{n!}\left(\frac{1}{2}\right)^n + R_{n+1} \,, \tag{252}$$

where the remainder is now

$$R_{n+1} = \frac{1}{(n+1)!}\left(\frac{1}{2}\right)^{n+1} \exp(\zeta) \tag{253}$$

for some $0 < \zeta < 1/2$.

Though we don't know what ζ actually, we can still estimate the worst case error. So for instance, if we wish to estimate $\exp(1/2)$ with a relative error of no more than 10^{-6}, how high does n have to be? The relative error (the ratio of the error to the exact result) associated with the nth order approximation is

$$\frac{R_{n+1}}{\exp(1/2)} = \frac{1}{(n+1)!}\left(\frac{1}{2}\right)^{n+1} \frac{\exp(\zeta)}{\exp(1/2)} \ . \tag{254}$$

Since $0 < \zeta < 1/2$, the biggest possible value of $\exp(\zeta)$ is $\exp(1/2)$, so from (254) the relative error is at worst

$$\frac{1}{(n+1)!}\left(\frac{1}{2}\right)^{n+1} \ . \tag{255}$$

Experimenting with a calculator shows that this maximum relative error is 1.55×10^{-6} for $n = 6$, and 9.69×10^{-8} for $n = 7$, so that to get a relative error of no more that 10^{-6} we require the seventh order approximation, $n = 7$.

Example 6.1 Find the nth order Taylor sum for $\exp x$ about $x = a$.

So we set $f(x) = e^x$, and note that $f^{(r)}(x) = e^x$ for all positive integers r. Thus $f^{(r)}(a) = e^a$. We next apply the formula:

$$f(x) = e^a + (x-a)e^a + \frac{1}{2}e^a(x-a)^2 + \cdots + \frac{1}{n!}e^a(x-a)^n + R_{n+1},$$

where the remainder term is given by

$$R_{n+1} = \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\zeta) = \frac{(x-a)^{n+1}}{(n+1)!}e^{\zeta},$$

and ζ is some number between a and x. Note that if x is much larger than a then the remainder might also be inconveniently big, unless of course n, the number of terms, is also sufficiently large.

6.3 Taylor series

If a function $f(x)$ is infinitely differentiable, and its remainder R_{n+1} goes to zero as $n \to \infty$, then we can continue adding successive terms to our Taylor polynomial *forever*. Ultimately we can represent $f(x)$ **exactly** as an *infinite series*, with the remainder term R_{n+1} dropped.

This is called a **Taylor series**. The Taylor series of $f(x)$ about $x = a$ is

$$\boxed{\begin{aligned} f(x) &= f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots \\ &\quad + \frac{(x-a)^n}{n!}f^{(n)}(a) + \ldots \,. \end{aligned}}$$

The special case of the Taylor series about $x = 0$ is called a **Maclaurin series**:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \ldots \,. \qquad (256)$$

Finally, we formally define the term **power series**. It refers to an infinite series where each term is proportional to a power of a variable x:

$$\sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots, \tag{257}$$

for some coefficients b_n. If a power series converges it is then necessarily the Taylor series of some function (cf. Borel's Theorem). Consequently, we will use the terminology power and Taylor series interchangeably.

6.4 Taylor series of the exponential function and its relatives

As a simple example, consider the power series expansion of $\exp x$ about $x = 0$. All the derivatives of $\exp(x)$ are equal to 1 at $x = 0$. Equation (256) then gives us

$$\boxed{\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.. + \frac{x^n}{n!} + \ldots. = \sum_{n=0}^{\infty} \frac{x^n}{n!}\ .} \tag{258}$$

By replacing x by $-x$ in (258) we find that

$$\boxed{\exp(-x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots.. + \frac{(-1)^n x^n}{n!} + \ldots. = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}\ ,} \tag{259}$$

and so

$$\cosh x = \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots. + \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots..\right]\right)\ . \tag{260}$$

The terms in even powers of x add together, but the odd terms cancel:

$$\boxed{\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} \ldots. + \frac{x^{2n}}{(2n)!} + \ldots. = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\ .} \tag{261}$$

Notice here how the general term is an even power of x.

In the same way, we can find the power series expansion for $\sinh x$:

$$\text{sinh}x = \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... - \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - .....\right]\right) . \tag{262}$$

The terms in odd powers of x now add together, while the even terms cancel:

$$\boxed{\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} .... + \frac{x^{2n+1}}{(2n+1)!} + ... = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} .} \tag{263}$$

Notice here how the general term is an odd power of x.

Example 6.2 Find the first three nonzero terms in the power series for $\tanh x$ about $x = 0$.

Set $f(x) = \tanh x$, and note that $f(0) = 0$. We next work out its derivatives evaluated at $x = 0$:

$$f'(x) = \text{sech}^2 x,$$
$$f''(x) = \frac{d(\cosh x)^{-2}}{dx} = -2\sinh x(\cosh x)^{-3},$$
$$f'''(x) = -2\cosh x(\cosh x)^{-3} + 6\sinh^2 x(\cosh x)^{-4},$$
$$f^{(4)}(x) = 4\sinh x(\cosh x)^{-3} + 12\sinh x(\cosh x)^{-3} - 24\sinh^3 x(\cosh x)^{-5},$$
$$f^{(5)}(x) = -48\sinh^2 x(\cosh x)^{-4} + 16(\cosh x)^{-2}$$
$$- 72\sinh^2 x(\cosh x)^{-4} + 120\sinh^4 x(\cosh x)^{-4},$$

with

$$f'(0) = 1, \quad f''(0) = 0, \quad , f'''(0) = -2, \quad f^{(4)}(0) = 0, \quad f^{(5)}(0) = 16.$$

If one still has one's sanity, we can then apply the formula to get the first three terms:

$$\begin{aligned}\tanh x &= x + \frac{1}{3!}(-2)x^3 + \frac{1}{5!}(16)x^5 + \ldots,\\ &= x - \frac{x^3}{3} + \frac{2x^5}{15} - \ldots.\end{aligned}$$

6.5 Power series of trigonometric functions

Consider $f(x) = \sin x$. Note that

$$\begin{aligned}\frac{\mathrm{d}\sin x}{\mathrm{d}x} &= \cos x &= \sin\left(x + \frac{\pi}{2}\right)\\ \frac{\mathrm{d}^2\sin x}{\mathrm{d}x^2} &= -\sin x &= \sin(x + \pi)\\ \frac{\mathrm{d}^3\sin x}{\mathrm{d}x^3} &= -\cos x &= \sin\left(x + \frac{3\pi}{2}\right)\\ \frac{\mathrm{d}^4\sin x}{\mathrm{d}x^4} &= \sin x &= \sin(x + 2\pi)\\ \vdots &= \vdots &= \vdots\end{aligned}$$

which leads us to the useful general result

$$\boxed{\frac{\mathrm{d}^n \sin x}{\mathrm{d}x^n} = \sin\left(x + \frac{n\pi}{2}\right)\ .} \tag{264}$$

Therefore

$$\begin{aligned} f(0) &= \sin(0) = 0 & f'(0) &= \sin(\pi/2) = 1 & f''(0) &= \sin(\pi) = 0 \\ f'''(0) &= \sin(3\pi/2) = -1 & f^{(4)}(0) &= \sin(2\pi) = 0....\ . \end{aligned}$$

Putting these results together:

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}..... + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\ .} \tag{265}$$

In the same way we can find a power series for $f(x) = \cos x$. Note that

$$\begin{aligned}
\frac{\mathrm{d}\cos x}{\mathrm{d}x} &= -\sin x &&= \cos\left(x + \frac{\pi}{2}\right) \\
\frac{\mathrm{d}^2\cos x}{\mathrm{d}x^2} &= -\cos x &&= \cos\left(x + \pi\right) \\
\frac{\mathrm{d}^3\cos x}{\mathrm{d}x^3} &= \sin x &&= \cos\left(x + \frac{3\pi}{2}\right) \\
\frac{\mathrm{d}^4\cos x}{\mathrm{d}x^4} &= \cos x &&= \cos\left(x + 2\pi\right) \\
\vdots &= \vdots &&= \vdots
\end{aligned}$$

which leads us to another useful general result

$$\boxed{\frac{\mathrm{d}^n\cos x}{\mathrm{d}x^n} = \cos\left(x + \frac{n\pi}{2}\right) \; .} \tag{266}$$

Therefore

$$\begin{aligned}
f(0) &= \cos(0) = 1 & f'(0) &= \cos(\pi/2) = 0 & f''(0) &= \cos(\pi) = -1 \\
f'''(0) &= \cos(3\pi/2) = 0 & f^{(4)}(0) &= \cos(2\pi) = 1 & &
\end{aligned} \tag{267}$$

Putting these results together

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}\ldots.. + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots. = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}} \tag{268}$$

Example 6.3 Find the power series for $\sin x$ about $x = a$.

Once again we set $f(x) = \sin x$, and note that the formula for the power series around $x = a$ is $f(x) = \sum_n (f^{(n)}(a)/n!)(x-a)^n$. From earlier

$$f^{(n)}(a) = \left.\frac{d^n \sin x}{dx^n}\right|_{x=a} = \sin\left(a + \frac{n\pi}{2}\right).$$

Therefore:

$$\sin x = \sum_{n=0}^{\infty} \frac{\sin\left(a + \frac{n\pi}{2}\right)}{n!}(x-a)^n,$$

$$= \sin a + \cos a\,(x-a) - \frac{\sin a}{2}(x-a)^2 - \frac{\cos a}{3!}(x-a)^3 + \frac{\sin a}{4!}(x-a)^4 + \ldots$$

An alternative way to this result is the following. Set $z = x - a$. Then

$$\sin x = \sin(z + a) = \sin z \cos a + \cos z \sin a.$$

Next we expand $\sin z$ and $\cos z$ around $z = 0$ using the canonical formulas (265) and (268), and reorganise the terms.

6.6 Euler's formula revisited

In Section 2.3 we made considerable use of the unproven result

$$\exp(i\theta) = \cos\theta + i\sin\theta \ . \tag{269}$$

Now we can prove it. First note from (258) that

$$\begin{aligned} \exp(\mathrm{i}\theta) &= 1 + \mathrm{i}\theta + \frac{(\mathrm{i}\theta)^2}{2!} + \frac{(\mathrm{i}\theta)^3}{3!} + \frac{(\mathrm{i}\theta)^4}{4!} + \frac{(\mathrm{i}\theta)^5}{5!} + \frac{(\mathrm{i}\theta)^6}{6!} + ... \\ &= 1 + \mathrm{i}\theta - \frac{\theta^2}{2!} - \frac{\mathrm{i}\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\mathrm{i}\theta^5}{5!} - \frac{\theta^6}{6!} + ... \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ... + \mathrm{i}\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...\right], \end{aligned} \tag{270}$$

which we now compare with the power series for $\cos\theta$ (equation 268) and the power series for $\sin\theta$ (equation 265).

It is clear that the real part of the right hand side of (270) is $\cos\theta$ and the imaginary part is $\sin\theta$, thereby proving (269).

6.7 Power series for logarithms

We can't derive a power series for $\ln x$ about $x = 0$, because $\ln 0$ is undefined (so the first term in (256) is undefined).

However, we can find a power series for $\ln(1 + x)$ about $x = 0$. Writing $f(x) = \ln(1 + x)$ we see that $f(0) = \ln 1 = 0$, while

$$\begin{array}{llllll} f'(x) = \frac{1}{1+x} & f''(x) = -\frac{1}{(1+x)^2} & f'''(x) = \frac{2}{(1+x)^3} \\ f^{(4)}(x) = -\frac{6}{(1+x)^4} & f^{(5)}(x) = \frac{24}{(1+x)^5} & f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n} . \end{array}$$

It follows that the r'th term in the power series is

$$\frac{f^{(n)}(0)}{n!}x^n = \frac{(-1)^{n-1}(n-1)!}{n!}x^n = \frac{(-1)^{n-1}}{n}x^n . \tag{271}$$

The power series for $\ln(1 + x)$ about $x = 0$ is therefore

$$\boxed{\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + ...(-1)^{n-1}\frac{x^n}{n} + ...} \tag{272}$$

By replacing x by $-x$ we can also find the power series for $\ln(1-x)$ about $x = 0$,

$$\ln(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \ldots + \frac{x^n}{n} + \ldots .\right] \tag{273}$$

The power series for $\sin x$, $\cos x$, $\exp x$ etc are valid for *any* real value of x. However, the power series for $\ln(1 \pm x)$ have a limited range of validity: the power series for $\ln(1+x)$ is valid on the real axis for $-1 < x \leqslant 1$, and the power series for $\ln(1-x)$ is valid on the real axis for $-1 \leqslant x < 1$.

6.8 The binomial expansion

We now consider the function

$$f(x) = (1+x)^{\alpha} ,$$

where α is a real number (not necessarily an integer).

By successive differentiation we find

$$\begin{aligned} f'(x) &= \alpha(1+x)^{\alpha-1} \\ f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2} \\ f'''(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3} \\ f^{(4)}(x) &= \alpha(\alpha-1)(\alpha-2)(\alpha-3)(1+x)^{\alpha-4} \\ \vdots &= \vdots \\ f^{(n)}(x) &= \alpha(\alpha-1)(\alpha-2)(\alpha-3)...(\alpha-n+1)(1+x)^{\alpha-n}\ . \end{aligned} \tag{274}$$

We can use this information to find the power series of $(1+x)^\alpha$ about $x = 0$:

$$\boxed{\begin{aligned} (1+x)^\alpha = \quad & 1 + \alpha x + \tfrac{\alpha(\alpha-1)}{2!}x^2 + \tfrac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 \\ & + \ldots \tfrac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)...(\alpha-n+1)}{n!}x^n + ... \end{aligned}} \tag{275}$$

The power series (275) is valid in the range $-1 < x < 1$ for general α.

In the special case when α is a positive integer, say $\alpha = N$, then the power series stops after a finite number of terms. Specifically, the coefficient of x^{N+1} is

$$\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)...(\alpha-N+1)(\alpha-N)}{(N+1)!}x^{N+1} , \tag{276}$$

but since $\alpha = N$ the final factor in (276) is precisely zero. Hence the term for x^{N+1} vanishes, as do the terms for any higher powers of x, since all such higher terms also contain the factor $\alpha - N$.

When $\alpha = N$ the power series (275) reduces to the polynomial:

$$\begin{aligned}(1+x)^N &= 1 + Nx + \frac{N(N-1)}{2!}x^2 + \frac{N(N-1)(N-2)}{3!}x^3 + ... \\ &+ \frac{N(N-1)(N-2)(N-3)...(1)}{N!}x^N .\end{aligned} \tag{277}$$

The general term in this sum is

$$\frac{N(N-1)(N-2)...(N-r+1)}{r!}x^r ,$$

which we can rearrange to be

$$\frac{N!}{(N-r)!r!}x^r \,, \tag{278}$$

or in other words the usual binomial coefficient. So when α is a positive integer (275) agrees with the familiar binomial expansion you have seen before:

$$(1+x)^N = \sum_{r=0}^{N} \binom{N}{r} x^r \,.$$

Finally, you will often encounter the alternative notations

$$\frac{N!}{(N-r)!r!} \equiv {}^N C_r \equiv C_r^N \equiv {}_N C_r \,. \tag{279}$$

<u>Example 6.4</u> Find the power series expansion about $x = 0$ of $(2+x)^{-1/2}$.

We re-express the function so it is easier to apply the binomial expansion:

$$(2+x)^{-1/2} = 2^{-1/2}\left(1+\frac{x}{2}\right)^{-1/2} .$$

Recall

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)\dots(\alpha-n+1)}{n!}x^n + \dots.$$

We see that $\alpha = -1/2$ and so we have:

$$(2+x)^{-1/2} = 2^{-1/2}\left[1 - \frac{1}{2}\left(\frac{x}{2}\right) + \frac{1}{2!}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\cdot\left(\frac{x}{2}\right)^2 + \frac{1}{3!}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\cdot\frac{-5}{2}\cdot\left(\frac{x}{2}\right)^3 + \dots\right],$$

$$= 2^{-1/2}\left[1 - \frac{x}{4} + \frac{3x^2}{32} - \frac{5x^3}{128} + \dots\right]$$

6.9 The Newton-Raphson method

We finish by outlining a method to approximately solve nonlinear algebraic equations such as $f(x) = 0$, where f is a nonlinear function.

- Suppose we have a rough guess for what the solution is, x_0, so that $f(x_0) \approx 0$. But we want to improve its accuracy. In other words, generate a new better approximation to the solution (call it x_1).
- We write $x_1 = x_0 + h$. We want to find h.
- So we set $f(x_1) = f(x_0 + h) = 0$ and then Taylor expand f around x_0 truncating at first order

$$0 = f(x_0 + h) \approx f(x_0) + hf'(x_0)$$

- We solve this approximate equation to get h and hence x_1, our better approximation:

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}} \tag{280}$$

- We can then repeat the procedure to get x_2, an even better approximation. And we continue this till we converge on to the exact solution (within some specified accuracy)

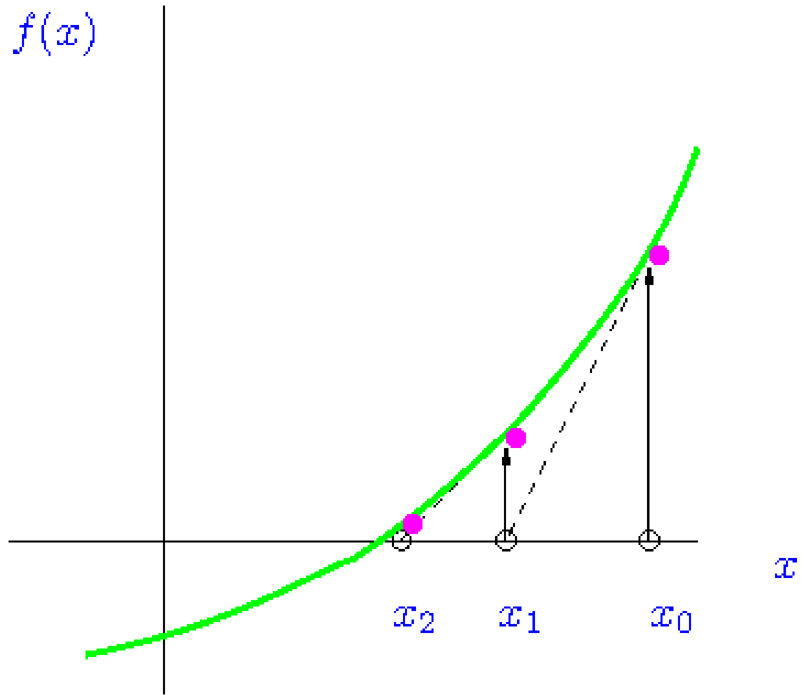


Figure 52: Successive approximations using the Newton-Raphson method.

Example: find the solution to $f(x) = x^2 - \ln(x + 5) = 0$.

Take as our initial guess $x_0 = 2$. Using a calculator, this yields $f(x_0) = 2.0541$, which is not very good. Let us see if the Newton-Raphson method can get us a better approximation:

i	x_i	$f(x_i)$	h
0	2.0	2.0541	-0.53254
1	1.4675	0.28665	-0.10310
2	1.3644	0.010758	-0.0041835
3	1.3602	1.7718×10^{-5}	-6.9125×10^{-6}

The Newton-Raphson method has a graphical interpretation, because solving $f(x) = 0$ is the same as finding the x-intercept of the curve $y = f(x)$.

Convergence of Newton-Raphson is very rapid when x_0 is near the solution. But if f has a turning point between x_0 and the exact solution then there is the danger that the method fails.

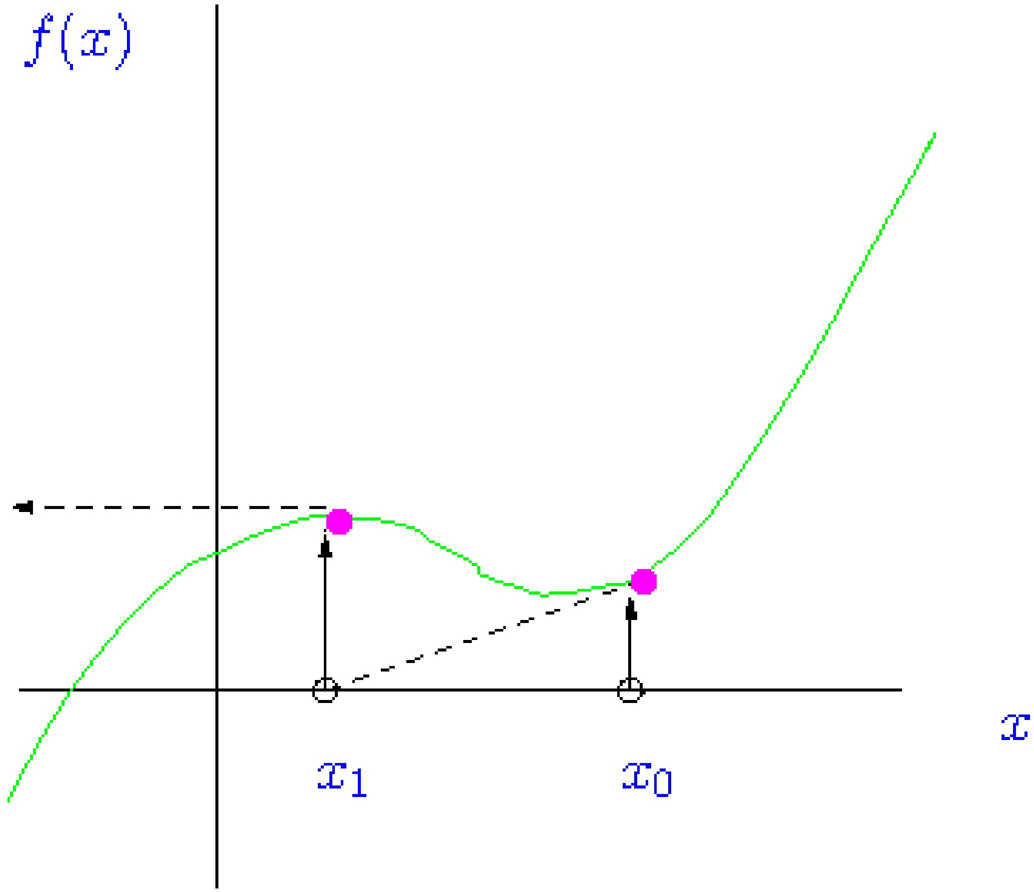


Figure 53: Example of when the Newton-Raphson may fail.

Example 6.5 [2004, paper 2, question 8D; 2006, paper 1, questions 9E]

(a) Find, by any method, the first three non-zero terms in the Taylor expansion about $x = 0$ of the following functions.

$$\frac{\log(1+x)}{1-x} \qquad \frac{1}{1+\sin x} \qquad \log(\cos x).$$

(b) Let

$$f(x) = \sum_{i=0}^{\infty} a_i x^i \quad \text{and} \quad g(x) = \sum_{i=0}^{\infty} b_i x^i$$

and let $\sum_{i=0}^{\infty} c_i x^i$ be the Taylor expansion about $x = 0$ of the function $f(x)g(x)$.

1. Find c_0, c_1 and c_2 in terms of a_0, a_1, a_2, b_0, b_1 and b_2.

2. Give a general expression for c_i as a finite sum of products of the coefficients a_j and b_j.

(c) Find the first four terms in the Taylor expansion around $x = 1$ of $\tan^{-1} x$

In (a) rather than work out all the complicated derivatives of these functions, let us write down the Taylor series of the component functions.

For the first one, recall

$$\ln(1 + x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \dots$$

and that

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \dots,$$

the infinite geometric series with ratio x (or the binomial expansion with $\alpha = -1$). In both expansions we are assuming that x is sufficiently small so that the series converge. Combining these two:

$$\begin{aligned}\frac{\ln(1+x)}{1-x} &= (x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots)(1 + x + x^2 + x^3 + \ldots),\\ &= x + x^2(1 - \tfrac{1}{2}) + x^3(\tfrac{1}{3} - \tfrac{1}{2} + 1) + \ldots,\\ &= x - \tfrac{1}{2}x^2 + \tfrac{5}{6}x^3 + \ldots.\end{aligned}$$

For the second one, introduce a new variable $y = \sin x$. If we are to expand around $x = 0$ then we expand around $y = 0$. Now

$$\frac{1}{1+\sin x} = \frac{1}{1+y} = 1 - y + y^2 - y^3 + \ldots,$$

the last equality coming about using either the binomial expansion with $\alpha = -1$ or recognising the infinite geometric series with ratio $-y$. (This series converges for all x except those values for which $\sin x = \pm 1$.) We next insert the Taylor series for $\sin x = x - x^3/3! + x^5/5! - \ldots$ and regroup the terms:

$$\begin{aligned}\frac{1}{1+\sin x} &= 1 - \sin x + (\sin x)^2 - (\sin x)^3 + \ldots \\ &= 1 - (x - x^3/3! + x^5/5! - \ldots) + (x - x^3/3! + x^5/5! - \ldots)^2 - \ldots \\ &= 1 - x + x^2 + \ldots\end{aligned}$$

Finally, for the third one, we play a similar trick and set $\cos x = 1 + z$, with the new variable z. Note that expanding around $x = 0$ means expanding around $z = 0$. First we have

$$\ln(\cos x) = \ln(1+z) = z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \ldots.$$

(This converges for sufficiently small z, hence sufficiently small x.) We next insert the Taylor series for $\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \ldots$, recognising that $z = -x^2/2! + x^4/4! - x^6/6! + \ldots$:

$$\begin{aligned}
\ln(\cos x) &= (-x^2/2! + x^4/4! - x^6/6! + \ldots) - \frac{1}{2}(-x^2/2! + x^4/4! - x^6/6! + \ldots)^2 \\
&\qquad + \frac{1}{3}(-x^2/2! + x^4/4! - x^6/6! + \ldots)^3 + \ldots, \\
&= -x^2/2 + x^4/24 - x^6/720 - \frac{1}{2}(x^4/4 - x^6/24 + \ldots) \\
&\qquad + \frac{1}{3}(-x^6/8 + \ldots) + \ldots, \\
&= -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} + \ldots
\end{aligned}$$

In part (b) we have a bonanza of terms to multiply ....

$$\begin{aligned}
f(x)g(x) &= (a_0 + a_1 x + a_2 x^2 + \ldots)(b_0 + b_1 x + b_2 x^2 + \ldots), \\
&= a_0 b_0 + (a_1 b_0 + a_0 b_1)x + (a_2 b_0 + a_1 b_1 + a_0 b_2)x^2 + \ldots \\
&= c_0 + c_1 x + c_2 x^2 + \ldots,
\end{aligned}$$

with $c_0 = a_0 b_0$, $c_1 = a_1 b_0 + a_0 b_1)$, and $c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2$.

We realise that the term proportional to x^l consists of a sum of all the products $a_i b_j x^{i+j}$ so that $i + j = l$. So we can write down the formula:

$$c_l = \sum_{i=0}^{l} a_i b_{l-i}.$$

Question (c) is a more straightforward one. Set $f(x) = \tan^{-1} x$. Then $f(1) = \pi/4$. And we have

$$f'(x) = \frac{1}{1+x^2},$$
$$f''(x) = \frac{-2x}{(1+x^2)^2},$$
$$f'''(x) = \frac{-2(1+x^2) - (-2x)4x}{(1+x^2)^3} = \frac{2(3x^2-1)}{(1+x^2)^3}.$$

Then $f'(1) = 1/2$, $f''(1) = -1/2$, and $f'''(1) = 1/2$, and finally:

$$\tan^{-1} x = \frac{\pi}{4} + \frac{x-1}{2} - \frac{(x-1)^2}{4} + \frac{(x-1)^3}{12} + \ldots .$$

7 Elementary Probability

Probability theory is essential in every field of science, even if it is not always used explicitly. It supplies a precise set of rules for exercising logic when we have incomplete information.

- Experiments and observations always involve random error (and usually a systematic error as well!), and so comparison with theory is inevitably probabilistic. Theory can only ever be proved up to a high level of significance.
- Some physical systems are inherently probabilistic, most notably in quantum mechanics, where we only ever compute probabilities because of the 'unknowable' nature of the subatomic world.
- Other systems, though formally deterministic, are so impossibly complicated that only a probabilistic or statistical description is feasible. Examples include:

chaotic dynamical systems, fluid turbulence, the kinetic theories of gases and of reaction rates in chemistry.

- Probability theory has many other applications: the social sciences; clinical trials, epidemiology, and public health; insurance; risk assessment in share trading and commodity markets; determining reliability in consumer products, such as cars and computers; etc.

- It is also useful when playing cards and other games of chance, which is why it was originally developed: in 1654, a dispute about a popular game of dice, between Pascal and Fermat, initiated the modern study of probability, and Abraham de Moivre's book on gaming in 1711 laid down its foundations.

Probability theory and statistics differ in the sense that probability deals with the likelihood of future events (given that we understand the underlying process creating them), while statistics takes as its task the analysis of previous events, in order to determine the underlying process giving rise to these events.

7.1 Basic concepts

7.1.1 Random experiments

We will be concerned with the outcomes of **random experiments**, that is, trials or observations which can be repeated many times but which contain an element of chance.

- **Outcomes:** The possible results of the experiments are called the *outcomes*. The outcomes must be *mutually exclusive* and we can label them (say) ω_1, ω_2, ... etc.

  For example, when throwing a six-sided die the outcomes are just the numbers from one to six, $\omega_1 = 1$, $\omega_2 = 2$, ..., $\omega_6 = 6$.

If the outcomes are described as "fair" or "unbiased" then they are equally likely.

- **Sample space:** The set of all possible outcomes of the experiment is called the *sample space*, $S = \{\omega_1, \omega_2, ...\}$.

  For the example of the die, $S = \{1, 2, 3, 4, 5, 6\}$.

- **Events:** An *event* A is a subset of the sample space S (so that $A \subset S$). An event may contain more than one outcome.

  For example, the event A might be 'throw an even number with the die', with $A = \{2, 4, 6\}$.

7.1.2 Elementary set theory

We are often concerned about whether or not two or more different events can happen together (simultaneously or consecutively) — and thus must deal with the relationship of two or more sets.

For the two events A and B we define the following sets:

1. $A \cap B$: the **intersection** of A and B, i.e. both events A and B occur;

2. $A \cup B$: the **union** of A and B, i.e. either event A, or event B, or both, occur.

3. $\overline{A}$: the **complement** of A, i.e. A does not occur. Other notations for the complement include A^c and A'.

4. $A - B$: outcomes in A which are not in B. Note that

$$A - B = A \cap \overline{B} \; . \tag{281}$$

A good way to represent these sets is via **Venn diagrams**.

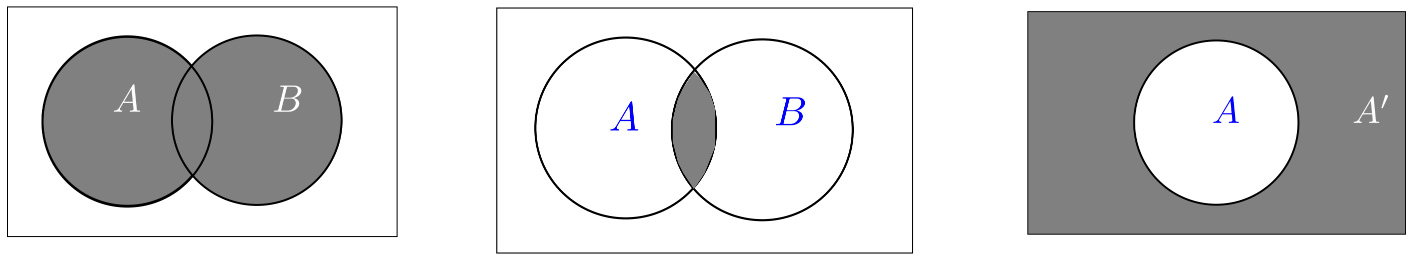


Figure 54: Panel 1 is $A \cup B$; panel 2 is $A \cap B$; lower panel is $\overline{A}$

The empty set, denoted $\varnothing$, contains no outcomes. Note that

$$A \cap \overline{A} = \varnothing, \qquad A \cup \overline{A} = S \ . \tag{282}$$

The events A and B are said to be **mutually exclusive** if they cannot both occur, i.e.

$$A \cap B = \varnothing \ . \tag{283}$$

7.1.3 Probability

The *probability* $P(A)$ expresses how likely an event A is. We will adopt the 'classical definition'.

Suppose in some experiment an event A corresponds to N_A specific outcomes, while the number of all possible outcomes is N. Then we define

$$P(A) = \frac{N_A}{N}. \tag{284}$$

It is assumed here that all outcomes are equally possible. For example, the probability of getting an even number when rolling a (fair) six-sided die is $3/6 = 1/2$.

The basic properties of probability are:

1. It is restricted between 0 and 1, i.e.

$$0 \leqslant P(A) \leqslant 1 \,, \tag{285}$$

with $P(S) = 1$ while $P(\varnothing) = 0$.

2. If $P(A) = 0$ then the event A is impossible.

3. For the complement of the event A we have

$$P(\overline{A}) = 1 - P(A) \ , \tag{286}$$

which is especially useful when $P(\overline{A})$ is easier to find than $P(A)$.

4. Additive rule for outcomes. Recall that the individual outcomes ω_1, ω_2, ... are mutually exclusive. So, if event $A_i = \{\omega_i\}$ for all i, and $A = \bigcup_i A_i$ is an event comprising some number of outcomes, then

$$P(A) = \sum_i P(A_i) \tag{287}$$

5. For the union of two general events A and B (that are not necessarily mutually

exclusive) we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)\ . \tag{288}$$

This result is easiest to see by drawing a Venn diagram - note that the term $P(A \cap B)$ is taken off the right of (288) to avoid double counting the region $A \cap B$.

If A and B are **mutually exclusive** then $P(A \cap B) = 0$ and

$$P(A \cup B) = P(A) + P(B)\,, \tag{289}$$

just as in point 4.

The result (288) can be extended to three events to give

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &- P(A \cap B) - P(B \cap C) - P(C \cap A) \\ &+ P(A \cap B \cap C)\ . \end{aligned} \tag{290}$$

This is best seen in a Venn diagram.

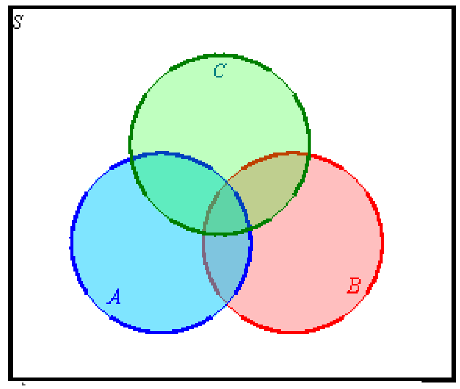


Figure 55: Venn diagram for $P(A \cup B \cup C)$.

<u>Example 7.1</u> A twelve-sided die is thrown. Event A is 'the number thrown is even', event B is the number thrown is 'divisible by three', and event C is 'the number thrown is one of $6, 7, 8, 9$'. Find the probabilities of $A \cap B$, $C \cap B$, $C \cap A$, $A - B$, $A \cap B \cap C$ and $\overline{A \cup B \cup C}$.

Let us first describe, in mathematical terms, the various events. We have:

$$A = \{2,\ 4\ ,6,\ 8,\ 10,\ 12\}, \quad B = \{3,\ 6,\ 9,\ 12\}, \quad C = \{6,\ 7,\ 8,\ 9\}.$$

And since we know there are 12 equally possible outcomes in a throw, we can write down:

$$P(A) = \frac{6}{12} = \frac{1}{2}, \qquad P(B) = \frac{1}{3}, \qquad P(C) = \frac{1}{3}.$$

Consider now the event defined by the intersection of A and B. Obviously $A \cap B = \{6,\ 12\}$. And thus its probability is $P(A \cap B) = 2/12 = 1/6$. Similarly $C \cap B = \{6,\ 9\}$ and $P(C \cap B) = 1/6$.

Now $A - B$ consists of all the outcomes in A that *are not* shared by B. Thus $A - B = \{2,\ 4,\ 8,\ 10\}$. And hence $P(A - B) = 1/3$.

The intersection of A, B, and C consists of only one outcome: $A \cap B \cap C = \{6\}$. Thus $P(A \cap B \cap C) = 1/12$. On the other hand the union of the three sets is

$$A \cup B \cup C = \{2,\ 3,\ 4,\ 6,\ 7,\ 8,\ 9,\ 10,\ 12\}$$

and comprises 9 outcomes. Hence $P(A \cup B \cup C) = 9/12 = 3/4$.

Finally, recognising that $P(\overline{D}) = 1 - P(D)$ for any event D, we can write $P(\overline{A \cup B \cup C}) = 1 - P(A \cup B \cup C) = 1 - 3/4 = 1/4$. This can be checked directly by noting that $\overline{A \cup B \cup C} = \{1, 5, 11\}$, thus comprising 3 events. Its probability is then $3/12 = 1/4$, in agreement with the other approach.

Example 7.2 A card is drawn at random from a pack. Event A is 'the card is an ace', event B is 'the card is a spade (♠)', event C is 'the card is one of ace, king, queen, jack, 10'. Calculate the probability that the card has (i) at least one of these properties; (ii) all of these properties.

So we can write

$$A = \{\text{ace of } \heartsuit, \text{ ace of } \diamondsuit, \text{ ace of } \clubsuit, \text{ ace of } \spadesuit\}$$

and note that there are 4 outcomes, while there are 52 cards in the pack. Hence $P(A) = 4/52 = 1/13$.

Event B consists of all the spades, of which there are 13 cards (2-10, plus the 3 royals, plus the ace). Therefore $P(B) = 13/52 = 1/4$.

Event C consists of 20 cards/outcomes, because each of the fours suits contains an ace, king, queen, jack, and 10. Thus $P(C) = 20/52 = 5/13$.

Part (i) asks us to find the probability of the union of these three events: $P(A \cup B \cup C)$. It is actually a tad easier to find $P(\overline{A \cup B \cup C})$ first. The event $\overline{A \cup B \cup C}$ consists of (a) no spades and (b) only the cards $2 - 9$ in the three suits of ♥, ♦, and ♣. This gives us $8 \times 3 = 24$ cards. Thus $P(\overline{A \cup B \cup C}) = 24/52 = 6/13$. Now we can answer the question:

$$P(A \cup B \cup C) = 1 - P(\overline{A \cup B \cup C}) = 1 - \frac{6}{13} = \frac{7}{13}.$$

Part (ii) asks us to find the intersection of these sets A, B, and C. We see straightaway that $A \cap B \cap C = \{\text{ace of } \spadesuit\}$. Thus $P(A \cap B \cap C) = 1/52$.

Example 7.3 A biased die has probability $p, 2p, 3p, 4p, 5p, 6p$ of throwing 1, 2, 3, 4, 5, 6 respectively. Find p. What is the probability of throwing an even number?

If S is the sample space, the total probability must be 1, i.e. $P(S) = 1$, or written out in full:

$$\sum_{n=1}^{6} P(\{n\}) = 1.$$

But each number rolled doesn't have an equal probability! Thus this can be rewritten

$$\sum_{n=1}^{6} np = p + 2p + 3p + 4p + 5p + 6p = 1.$$

We can solve this equation for p, and find that $p = 1/21$.

Let us denote by A the event that we roll an even number, and so $A = \{2, 4, 6\}$.

We can then work out its probability directly by looking at each of the outcomes that it consists of:

$$P(A) = P(\{2\}) + P(\{4\}) + P(\{6\}) = 2p + 4p + 6p = 12p = \frac{4}{7}.$$

7.2 Conditional probability

We are often interested in determining the probability of one event given that another, possibly related, event has occurred.

The probability that event A occurs, given that event B has occurred, is denoted $P(A|B)$ and is called the **conditional probability**.

For example, event A might correspond to a student getting a first, while B corresponds to a student attending every lecture. $P(A|B)$ is the probability that a student gets a first given that they attended every lecture. $P(A|B)$ is presumably different to $P(A)$, the *unconditional* probability that a student gets a first.

Because the event B is known to have occurred, then the event B replaces S as the sample space when we try to compute the event $B|A$. This then motivates

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \ , \tag{291}$$

where we have had to normalise the probability of the intersection by $P(B)$.

- If two events A and B are mutually exclusive then we have $P(A|B) = P(B|A) = 0$,

  e.g. if one card is drawn from a pack, and event A is that a red card is drawn while event B is that a club (♣) card is drawn, then B and A are mutually exclusive.

- We say two events A and B are *independent* if $P(A|B) = P(A)$,

  e.g. event B is tossing a coin and getting heads; event A is tossing the coin a second time and getting tails.

- Be careful, usually $P(A|B) \neq P(B|A)$.

Example: we will find the probability that the single throw of a fair die results in a number less than four, given that the throw resulted in an odd number. Let B be the event of the throw being less than four, i.e. $B = \{1, 2, 3\}$, then it follows that

$$P(B) = \frac{3}{6} = \frac{1}{2}\,. \tag{292}$$

Let A be the event that the throw of the die is an odd number, i.e. $A = \{1, 3, 5\}$, then $P(A) = 1/2$ as well.

The event $B \cap A$ is a throw which is less than 4 **and** odd, i.e. $B \cap A = \{1, 3\}$, so that

$$P(B \cap A) = \frac{2}{6} = \frac{1}{3}\,. \tag{293}$$

It then follows from (291) that

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{1/2} = \frac{2}{3}\,. \tag{294}$$

<u>Example 7.4</u> Consider drawing 2 balls out of a bag of 5 balls: 1 red, 2 green, 2 blue. What is the probability of drawing a blue ball out of the bag given that the first ball was blue if: (i) the first ball is replaced; (ii) the first ball is not replaced.

Part (i) is relatively easy. If the first blue ball is replaced then there are two blue balls in a bag of five balls, and hence the probability of drawing one of blues is $2/5$. There is no ‘memory‘ of the first draw because the first ball was replaced. And thus there is no need to use the concepts of conditional probability here.

Part (ii), however, does require us to use conditional probability, because the first draw has impacted on subsequent draws. Let us define the following events: event A denotes getting a blue ball in the first draw. Thus $P(A) = 2/5$ (there are two blue balls out of five); event B denotes getting a blue ball in the second draw. The question asks us to find $P(B|A)$.

The event that both balls drawn are blue is just $B \cap A$, and its probability is simply $P(B \cap A) = (2/5) \times (1/4) = 1/10$, i.e. the product of getting a blue in the first draw (2/5) and of getting a blue in the second (1/4). The conditional probability (which is different to the probability of the intersection!), can then be computed

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/10}{2/5} = \frac{1}{4}.$$

7.2.1 Law of total probability

This law expresses the total probability of an outcome if it can be realized via several distinct events. Consider two events A and B. The law states that

$$P(B) = P(B|A)P(A) + P(B|\overline{A})P(\overline{A}) \ . \tag{295}$$

This holds because there are **two** distinct possible routes to getting event B.

1. The first route is: event A (probability $P(A)$) happens, and then B happens (probability $P(B|A)$). The total probability for this route is

$$P(A) \times P(B|A) \; . \tag{296}$$

2. The second route is: event A *does not* happen (probability $P(\overline{A})$), and then B happens (probability $P(B|\overline{A})$). The total probability for this route is

$$P(\overline{A}) \times P(B|\overline{A}) \; . \tag{297}$$

Adding together the probabilities in (296) and (297) gives (295).

7.2.2 Bayes' Theorem

We now come to the most important result in conditional probability, known as Bayes' Theorem.

The probability of A occurring given B is $P(A|B)$, cf. equation (291). On the other hand, the probability of event B occurring given that A has already occurred is $P(B|A)$, with

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \,. \tag{298}$$

We then have

$$\begin{aligned} P(A \cap B) &= P(B|A)P(A) \\ P(B \cap A) &= P(A|B)P(B) \,. \end{aligned} \tag{299}$$

Furthermore, we know that $P(A \cap B) = P(B \cap A)$, and using (299) this leads to

$$\boxed{P(A|B) = \frac{P(A)P(B|A)}{P(B)} \,,} \tag{300}$$

provided that $P(B) \neq 0$. This is Bayes' Theorem.

Using the law of total probability (300) gives the alternative form of Bayes' Theorem:

$$\boxed{P(A|B) = \frac{P(A)P(B|A)}{P(B|A)P(A) + P(B|\overline{A})P(\overline{A})}} \; . \tag{301}$$

This is sometimes easier to use than (300).

We consider the following example:

A screening test is 99% effective in detecting a certain disease when a person has the disease. The test yields a 'false positive' for 1% of healthy persons tested. If 0.1% of the population have the disease then what is the probability that a person whose test is positive has the disease?

Let the event A be that a person has the disease, so that $P(A) = 0.001$, and the probability that they do not have the disease is $P(\overline{A}) = 1 - P(A) = 0.999$. Let the event B be a positive test, so that $P(B|A) = 0.99$ (i.e. the probability of the test successfully detecting the disease) and $P(B|\overline{A}) = 0.01$ (i.e. the probability of a positive test on a healthy person).

From (301) we therefore have

$$\begin{aligned} P(A|B) &= \frac{P(A)P(B|A)}{P(B|A)P(A) + P(B|\overline{A})P(\overline{A})} \\ &= \frac{0.001 \times 0.99}{0.99 \times 0.001 + 0.01 \times 0.999} \\ &= 0.0902 \end{aligned}$$

or roughly 9%. So the probability of someone who tests positive actually having the disease is rather low! This has arisen because the probability of a false positive, 0.01, is large compared to the probability of having the disease, 0.001.

Example 7.5 Calculate the probability that someone who tests negative actually has the disease after all, $P(A|\overline{B})$.

So we apply Bayes' Theorem again so that

$$P(A|\overline{B}) = \frac{P(A)P(\overline{B}|A)}{P(\overline{B}|A)P(A) + P(\overline{B}|\overline{A})P(\overline{A})}.$$

To find $P(\overline{B}|A)$ we note that $P(\overline{B}|A) + P(B|A) = 1$, since if A happens there are only two mutually exclusive possibilities B and $\overline{B}$. We know that $P(B|A) = 0.99$, therefore $P(\overline{B}|A) = 1 - P(B|A) = 1 - 0.99 = 0.01$.

Similarly, $P(\overline{B}|\overline{A}) + P(B|\overline{A}) = 1$, and thus $P(\overline{B}|\overline{A}) = 1 - P(B|\overline{A}) = 1 - 0.01 = 0.99$.

Putting everything together now, we get

$$P(A|\overline{B}) = \frac{0.001 \times 0.01}{0.001 \times 0.01 + 0.99 \times 0.999} = \frac{0.00001}{0.98901} \approx 10^{-5},$$

which is quite a small probability.

7.3 Combinatorics

We often worry about problems which involve completing a sequence of actions or arranging/grouping elements, e.g. tossing a coin 10 times, drawing coloured balls from a bag, or arranging N indistinguishable particles into R energy levels (Bose-Einstein or Fermi-Dirac statistics).

When the order of the actions/elements matters, then we are dealing with *permutations*.

When the order of the actions/elements does not matter, then we are dealing with *combinations*.

7.3.1 Permutations

Consider n distinguishable objects, and let us select r of them without replacement and put them in order. This is a permutation.

How many different permutations are possible?

- The first time we select an object, there are n possible choices.
- The second time there are only $n-1$ choices
- The third time there are only $n-2$ choices, and on and on until we have taken r objects.

It follows that the total number of possibilities is the product of all the choices

$$n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!} \equiv {}^{n}P_{r} \; . \tag{302}$$

We have that ${}^{n}P_{n} = n!$, since $0! \equiv 1$.

A simple example: how many ways can all the club (♣) cards be arranged in a line?

There are $n = 13$ cards to draw from. And we are going to arrange all of them, so $r = 13$. Therefore the number of permutations is

$$^{13}P_{13} = 13! = 6,227,020,800.$$

So quite a few.

How about the number of ways to order 4 cards from the club suite?

That would be $^{13}P_4 = 13!/9! = 17,160$.

Example 7.6 What is the probability that in a room of N people at least 2 have the same birthday? Take $N = 200$.

First, let us take the number of days in the year to be 365, for simplicity. Next, let A denote the event that two or more people have the same birthday. It will turn out, however, to be easier to work with its complement $\overline{A}$: the event that everyone in the room has a different birthday. Consider the number of possible permutations of *different* birthdays amongst the N people: in other words, how many lists of N different numbers can be made out of a choice of 365? This is equal to

$$^{365}P_N = 365 \times 364 \times 363 \times \cdots \times (365 - N + 1) = \frac{365!}{(365-N)!}.$$

On the other hand, the total number of different birthday distributions amongst the N people (allowing repetitions) is just

$$365 \times 365 \times 365 \times \cdots \times 365 = 365^N.$$

The probability $P(\overline{A})$ is the division of the first number by the second:

$$P(\overline{A}) = \frac{365!}{(365-N)!365^N}.$$

Now we can compute $P(A)$, via

$$P(A) = 1 - P(\overline{A}) = 1 - \frac{365!}{(365-N)!365^N}.$$

As N gets large the probability rapidly approaches 1. For example, if $N = 23$, then $P(A) \approx 1/2$. If $N = 50$, then $P(A) \approx 0.97$. Finally, when $N = 200$, we get $P(A) \approx 1 - 10^{-30} \approx 1$. In the last case it is 'guaranteed' that at least two people share a birthday.

7.3.2 Combinations

Let us, as before, select r objects from a set of n, but let us not care about the ordering of the selection. (For example, how many hands of 5 cards are there from a full deck of 52 cards?) This is a *combination*. How many different combinations are there?

- So there are still ${}^nP_r$ ordered arrangements, as before, but we do not care about the ordering.

- Each group of r objects can be ordered $r!$ ways, but we do not want to count all of these different orderings.

- Hence we divide through ${}^nP_r$ by $r!$ to get the number of combinations:

$$\frac{{}^nP_r}{r!} = \frac{n!}{(n-r)!r!} \equiv {}^nC_r \ . \tag{303}$$

As an example, suppose we draw that hand of 5 cards from a deck of 52 cards. The number of different hands possible is

$${}^{52}C_5 \equiv \frac{52!}{(52-5)!5!} = \frac{52 \times 51 \times 50 \times 49 \times 48}{120} = 2,598,960 \ . \tag{304}$$

Example 7.7 Out of 10 physics professors and 12 chemistry professors, a committee of 5 people must be chosen in which each subject has at least 2 representatives. In how many ways can this be done?

First thing to note is that there are going to be two kinds of committee: one with 3 physicists plus 2 chemists, and one with 2 physicists plus 3 chemists. We deal with each kind separately.

The first kind of committee has 3 physicists, which must be chosen from a pool of 10. But the ordering doesn't matter. Therefore the number of ways of choosing the 3 is ${}^{10}C_3 = 10 \times 9 \times 8/6 = 120$. On the other hand, the number of ways of choosing the 2 chemists from a pool of 12 is ${}^{12}C_2 = 12 \times 11/2 = 66$. Putting this information together, the total number of different committees of the first kind is $120 \times 66 = 7920$.

The second kind of committee has only 2 physicists. The number of ways of choosing these 2 is ${}^{10}C_2 = 45$. The number of ways of choosing 3 chemists is ${}^{12}C_3 = 220$. Therefore the total number of different committees of the second kind is $45 \times 220 = 9900$.

Finally, the total number of committees of the two kinds allowed = $7920 + 9900 = 17820$.

Binomial coefficients

The numbers ${}^{n}C_r$ are called the binomial coefficients, because they arise in the binomial expansion. In particular,

$$\boxed{(p+q)^n = \sum_{r=0}^{n} {}^nC_r q^r p^{n-r} \ .} \tag{305}$$

They have the interesting property that

$${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1} \ . \tag{306}$$

To prove this recursion relation note that

$$\begin{aligned} {}^{n-1}C_r + {}^{n-1}C_{r-1} &= \frac{(n-1)!}{r!(n-1-r)!} + \frac{(n-1)!}{(r-1)!(n-r)!} \\ &= \frac{(n-1)!(n-r)+(n-1)!r}{r!(n-r)!} \\ &= \frac{(n-1)!(n-r+r)}{r!(n-r)!} = \frac{(n-1)!n}{r!(n-r)!} \\ &= \frac{n!}{r!(n-r)!} = {}^nC_r \ . \end{aligned} \tag{307}$$

7.4 Random variables

A **random variable** X is a variable whose value depends on the outcomes of an experiment involving some level of randomness or chance.

A random variable may take discrete values (e.g. the outcome of coin tosses) or it may take a range of continuous values (e.g. the mass of newborn babies).

- For example, let us toss a coin 3 times (our experiment) and take X to represent the number of heads (the random variable).

  The sample space of outcomes is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

  and the value of X can be 0, 1, 2 or 3. In this case, the variable X is **discrete** because it can only take discrete values 0,1,2,3. Note that more than one outcome can give the same value of the random variable.

- A different example: consider an aircraft flying through some particularly nasty clear-air turbulence that causes its velocity $\boldsymbol{v}$ to randomly vary. If we take X to be the instantaneous speed of the plane $|\boldsymbol{v}|$ we see that X can take **continuous** values between 0 and ∞, in principle. Again different outcomes can yield the same X (e.g. a sharp deviation up and a sharp and equal deviation down both generate the same speed).

7.5 Discrete probability distributions

Consider a discrete random variable X. For each value X takes, we can assign a probability.

If X takes the discrete values x_i, which have associated probabilities p_i, for $i = 1, 2, ..n$, then $P(X = x_i) = p_i$.

We can then construct a **probability function**, also called a **probability dis-**

tribution, usually just denoted $P(X)$, which is simply the probability of any event associated with X in S.

This function is normalised, as expected, according to

$$\sum_{i=1}^{n} P(X = x_i) = \sum_{i=1}^{n} p_i = 1 \ . \qquad (308)$$

As an example, let us return to the coin-tossing game earlier:

- $X = 0$ corresponds only to TTT, so $P(X = 0) = 1/8$,
- $X = 1$ corresponds to the event $\{HTT, THT, TTH\}$, so $P(X = 1) = 3/8$
- $X = 2$ yields $P(X = 2) = 3/8$, and $X = 3$ yields $P(X = 3) = 1/8$.

The **cumulative probability function** (CPF), $F(x)$, is the probability that X takes a value which is less than or equal to x, i.e.

$$F(x) = P(X \leqslant x) = \sum_{x_i \leqslant x} P(X = x_i) \ . \qquad (309)$$

So for a discrete random variable the CPF will be a series of steps at each value x_i, and will be constant between the steps. Note how the ultimate value of $F(x) \to 1$ as $x \to \infty$ (in the coin tossing game, $x \geqslant 3$).

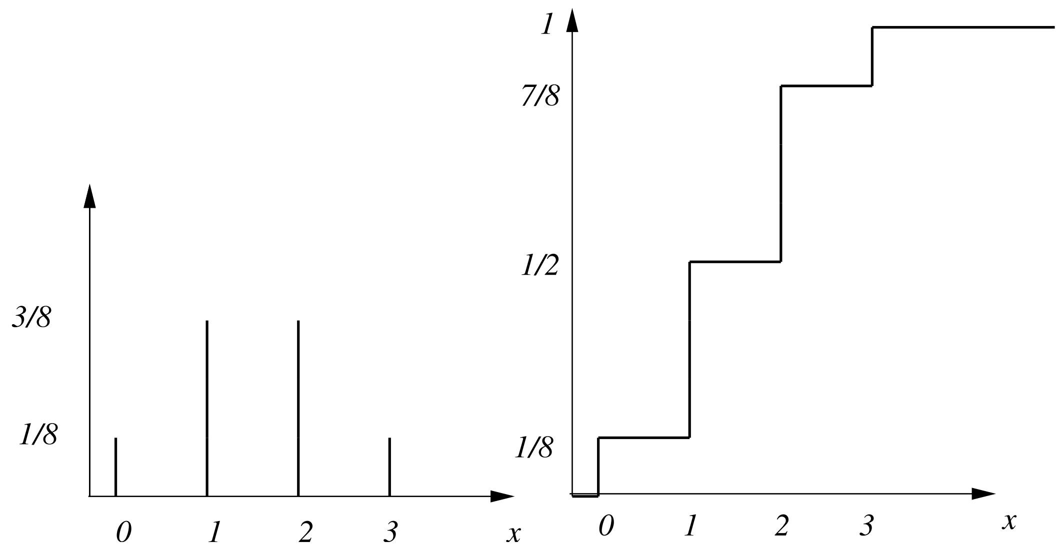


Figure 56: Graphs of $P(X)$ and $F(x)$ for the coin-tossing example.

Example 7.8 A bag contains 6 blue balls and 4 red balls. Three balls are drawn without replacement. Find the probability function for the number of red balls drawn.

Let X be our random variable, corresponding to the number of red balls drawn. We now construct the probability distribution $P(X)$ piece by piece, by looking at $X = 0, 1, 2$ and 3 separately.

- We have $X = 3$ when we draw a red on each draw, i.e. 'RRR'. The probability of this is $4/10 \times 3/9 \times 2/8$, which is equal to 1/30. Thus $P(3) = 1/30$.
- Three different kinds of draw yield $X = 2$: RRB, RBR, and BRR (where B denotes a blue draw). The probability of RRB is $4/10 \times 3/9 \times 6/8 = 1/10$. The probability of RBR is $4/10 \times 6/9 \times 3/8 = 1/10$. The probability of BRR is $6/10 \times 4/9 \times 3/8 = 1/10$. Thus $P(2) = 1/10 + 1/10 + 1/10 = 3/10$.
- There are also three different kinds of draw to give us $X = 1$: RBB, BRB, and BBR. The probability of RBB is $4/10 \times 6/9 \times 5/8 = 1/6$. The probability

of BRB is also 1/6, as is the probability of BBR. Therefore $P(1) = 1/6 + 1/6 + 1/6 = 1/2$.

- Finally, we examine $X = 0$, which corresponds only to one kind of draw: BBB. Its probability is $6/10 \times 5/9 \times 4/8 = 1/6$.

In summary, our probability distribution $P(X)$ is defined through:

$$P(3) = \frac{1}{30}, \quad P(2) = \frac{3}{10}, \quad P(1) = \frac{1}{2}, \quad P(0) = \frac{1}{6}.$$

Just to check that it is normalised appropriately, we look at the sum

$$\sum_{X=0}^{3} P(X) = \frac{1}{6} + \frac{1}{2} + \frac{3}{10} + \frac{1}{30} = \frac{5 + 15 + 9 + 1}{30} = 1,$$

and all is good.

7.5.1 Mean and Variance

The **mean** of the random variable X is defined to be

$$\boxed{\mathbb{E}[X] \equiv \sum_{i=1}^{n} x_i p_i = \sum_{i=1}^{n} x_i P(X = x_i)\ .} \tag{310}$$

The mean is also often referred to as the **expectation value**, and the alternative notations $E[X]$, $\langle X \rangle$, $\overline{X}$, or μ are often used.

If an experiment is repeated a very large number of times then the average value of the associated random variable X will approach the mean (cf. the Law of Large Numbers).

For example, consider the three coin tosses in the previous subsection:

$$\begin{aligned}\mathbb{E}[X] &= \sum_{i=1}^{n} x_i P(X = x_i) \\ &= 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} \\ &= \frac{3}{2}\,.\end{aligned}$$

The mean has the following properties:

1. $\mathbb{E}[aX] = a\mathbb{E}[X]$, where a is a constant;

2. If X and Y are two different random variables (possibly with different probability functions), then

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]\,; \tag{311}$$

3. If $g(X)$ is a function of the random variable X then

$$\mathbb{E}[g(X)] = \sum_{i=1}^{n} g(x_i) P(X = x_i) \ . \tag{312}$$

We are often interested in the way in which results are spread around the mean. One measure of this is the **variance** of X, which we define to be

$$\boxed{\text{var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] \ .} \tag{313}$$

In other words, the variance is the mean value of the square of the deviation from the mean.

The **standard deviation** σ is the square root of the variance, i.e.

$$\boxed{\sigma^2 = \text{var}(X)}. \tag{314}$$

Expanding the bracket in (313) leads to

$$
\begin{aligned}
\sigma^2 &= \mathbb{E}[X^2 - 2X\mu + \mu^2] \\
&= \mathbb{E}[X^2] - \mathbb{E}[2\mu X] + \mathbb{E}[\mu^2] \\
&= \mathbb{E}[X^2] - 2\mu\mathbb{E}[X] + \mu^2 \quad \text{(since } \mu \text{ is a constant)} \\
&= \mathbb{E}[X^2] - 2\mu\mu + \mu^2 \\
&= \mathbb{E}[X^2] - \mu^2 \; . \qquad (315)
\end{aligned}
$$

From (315) we are therefore left with the very useful result that the variance σ^2 and the mean μ are related by

$$\boxed{\sigma^2 = \mathbb{E}[X^2] - \mu^2 \; .} \qquad (316)$$

Note that in this expression:

$$\mathbb{E}[X^2] = \sum_{i=1}^{n} x_i^2 P(X = x_i) \; . \qquad (317)$$

For our experiment of tossing three coins with X being the number of heads, we have

$$\begin{aligned}\mathbb{E}[X^2] &= 0^2 \times \frac{1}{8} + 1^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{8} \\ &= 3\,,\end{aligned}$$

so the variance is given as

$$\sigma^2 = \mathbb{E}[X^2] - \mu^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}\,. \tag{318}$$

7.5.2 Binomial distribution

The binomial distribution arises when an experiment has only 2 possible outcomes, e.g. a single coin toss (heads or tails), probability of surviving a heart attack (yes or no), asking a random person if they can drive a tractor (yes or no).

- Let event A, labeled a 'success', denote one outcome and $B = \overline{A}$, labeled a 'failure', the other outcome.
- If the probability of event A is p then the probability of event B is $q = 1 - p$.
- Suppose that the experiment is repeated n times, and let the discrete random variable X be the number of successes, so that X takes one of the values $0, 1, 2, ..., n$. The probability distribution $P(X)$ may be written as

$$P(X = r) \equiv B_r(n, p) = {}^nC_r p^r (1 - p)^{n-r} \ . \qquad (319)$$

  This is the **binomial distribution**.

How did we get to the second formula?

- The sample space of n experiments contains 2^n outcomes.
- Each outcome that has r successes has a probability $p^r q^{n-r}$.

- But there are lots of outcomes with $X = r$. We need to add them all up to get our correct probability:

$$P(X = r) = \overbrace{p^r q^{n-r} + p^r q^{n-r} + \cdots + p^r q^{n-r}}^{M} = M \cdot p^r q^{n-r}$$

  How many of them are there? That is, what is M?

- Actually, M is simply the number of ways of choosing r elements from a set of n, i.e. ${}^nC_r$.

We can check that the binomial distribution satisfies the normalisation condition (308).

First recall that

$$(p + q)^n = \sum_{r=0}^{n} {}^nC_r q^r p^{n-r},$$

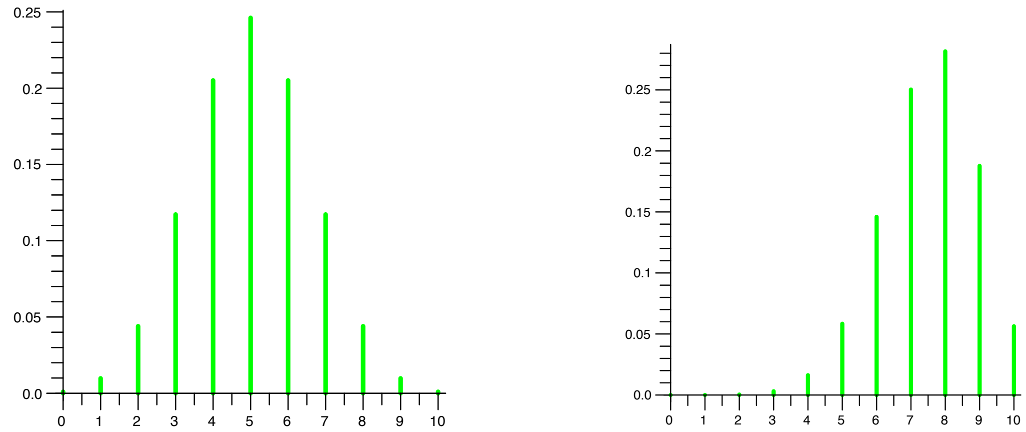


Figure 57: Binomial distributions: $B(10, 0.5)$ (left) and $B(10, 0.75)$ (right).

which gives

$$\sum_{r=0}^{n} P(X = r) = \sum_{r=0}^{n} {}^{n}C_{r} p^{r} (1-p)^{n-r} = (p + 1 - p)^{n} = 1 \ . \tag{320}$$

Mean

We can calculate the mean of the binomial distribution:

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{r=0}^{n} r \left[{}^{n}C_r p^r (1-p)^{n-r} \right] \\
&= \sum_{r=0}^{n} r \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r} \\
&= \sum_{r=1}^{n} \frac{n!}{(r-1)!(n-r)!} p^r (1-p)^{n-r} \quad \text{(the } r = 0 \text{ term is zero)}
\end{aligned}$$

$$
\begin{aligned}
&= n\sum_{r=1}^{n}\frac{(n-1)!}{(r-1)!(n-r)!}p^{r}(1-p)^{n-r} \\
&= n\sum_{s=0}^{n-1}\frac{(n-1)!}{s!(n-1-s)!}p^{s+1}(1-p)^{n-1-s} \quad \text{(writing } r = s+1\text{)} \\
&= np\sum_{s=0}^{n-1}\frac{(n-1)!}{s!(n-1-s)!}p^{s}(1-p)^{n-1-s} \\
&= np(1-p+p)^{n-1} = np\ . \qquad (321)
\end{aligned}
$$

Recall that

$$
(p+q)^{n} = \sum_{r=0}^{n} {}^{n}C_{r}q^{r}p^{n-r},
$$

So the mean of the binomial distribution is $\mu = np$.

Variance

To find the variance of the binomial distribution we first write

$$\begin{aligned}
\mathbb{E}(X^2) &= \sum_{r=0}^{n} r^2 \left[{}^nC_r p^r (1-p)^{n-r} \right] \\
&= \sum_{r=0}^{n} r(r-1) \left[{}^nC_r p^r (1-p)^{n-r} \right] + \sum_{r=0}^{n} r \left[{}^nC_r p^r (1-p)^{n-r} \right] \quad (322)
\end{aligned}$$

The second term is exactly $\mathbb{E}(X)$ (see first line of 321), and is therefore np.

The first term in (322) is

$$
\begin{aligned}
&= \sum_{r=0}^{n} r(r-1)\frac{n!}{r!(n-r)!}p^{r}(1-p)^{n-r} \\
&= \sum_{r=2}^{n} \frac{n!}{(r-2)!(n-r)!}p^{r}(1-p)^{n-r} \quad \text{(the } r = 0,1 \text{ terms are zero)} \\
&= n(n-1)p^{2}\sum_{r=2}^{n} \frac{(n-2)!}{(r-2)!(n-r)!}p^{r-2}(1-p)^{n-r} \\
&= n(n-1)p^{2}\sum_{s=0}^{n-2} \frac{(n-2)!}{(s)!(n-2-s)!}p^{s}(1-p)^{n-2-s} \quad \text{(writing } r = s+2\text{)} \\
&= n(n-1)p^{2}(1-p+p)^{n-2} = n(n-1)p^{2} \; . \qquad (323)
\end{aligned}
$$

Hence, back in (322) we have

$$\mathbb{E}(X^{2}) = n(n-1)p^{2} + np \; , \qquad (324)$$

and using (316) and (322) we have the variance of the binomial distribution as

$$\sigma^2 = n(n-1)p^2 + np - (np)^2 = np(1-p) \; . \tag{325}$$

So the standard deviation of the binomial distribution is $\sigma = \sqrt{np(1-p)}$. Interestingly, as the sample size, n, gets larger the mean of X increases like n but the standard deviation, σ, increases less rapidly, like $\sqrt{n}$. Thus the relative width of the distribution $\sigma/\mu \propto 1/\sqrt{n}$, and it gets narrower and narrower the more experiments we do.

Thus many experimental measurements suppress random errors.

7.5.3 Poisson distribution

The Poisson distribution arises when the number of 'successes' in a given event is unlimited. For instance, what is the distribution of the number of photons received by a telescope per minute? How many golfers are struck by lightning every year? The number of goals scored in a football match?

If X is the number of successes (photons received, lightning strikes, etc.), then it turns out that

$$\boxed{P(X=r)=\frac{\lambda^r\exp(-\lambda)}{r!}\,,} \tag{326}$$

which is the **Poisson distribution** with parameter λ. Here, r is the number of event occurrences, taking possible values $r = 0, 1, 2, 3, \ldots.$ right up to infinity.

The Poisson distribution is derived as follows. Consider the binomial distribution

$$P(X=r)=B_r(n,p)={}^nC_r p^r(1-p)^{n-r}.$$

Let $\lambda = np$ (the mean) and consider $B_r(n, \lambda/n)$. Then

$$\lim_{n\to\infty} B_r(n,\lambda/n)=\frac{\lambda^r\exp(-\lambda)}{r!}.$$

See Figure 59.

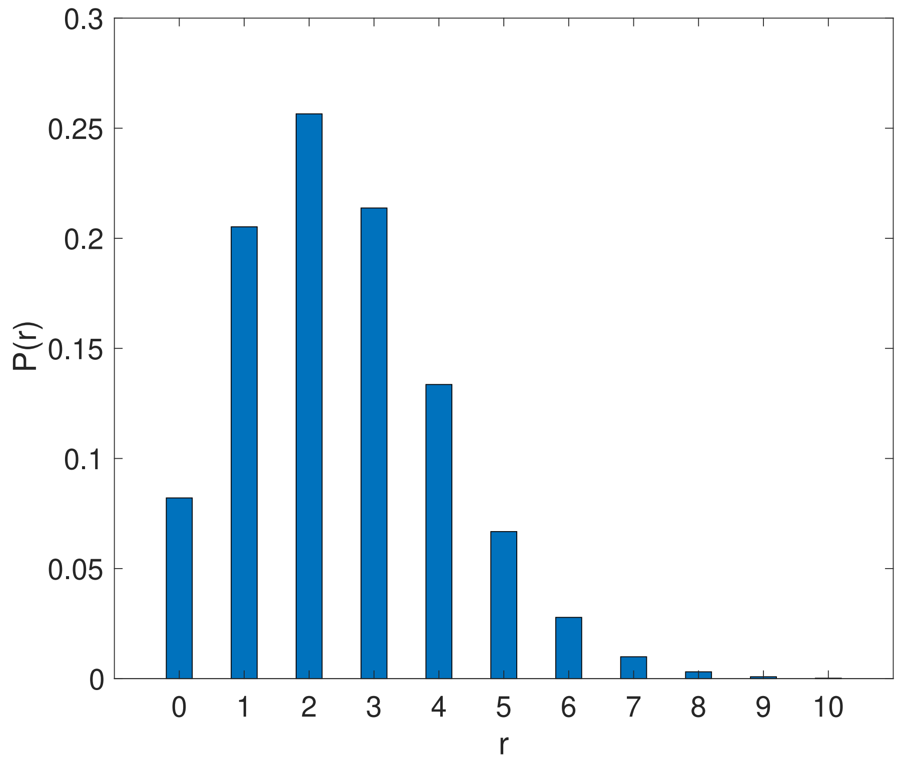


Figure 58: Poisson distribution with $\lambda = 2.5$

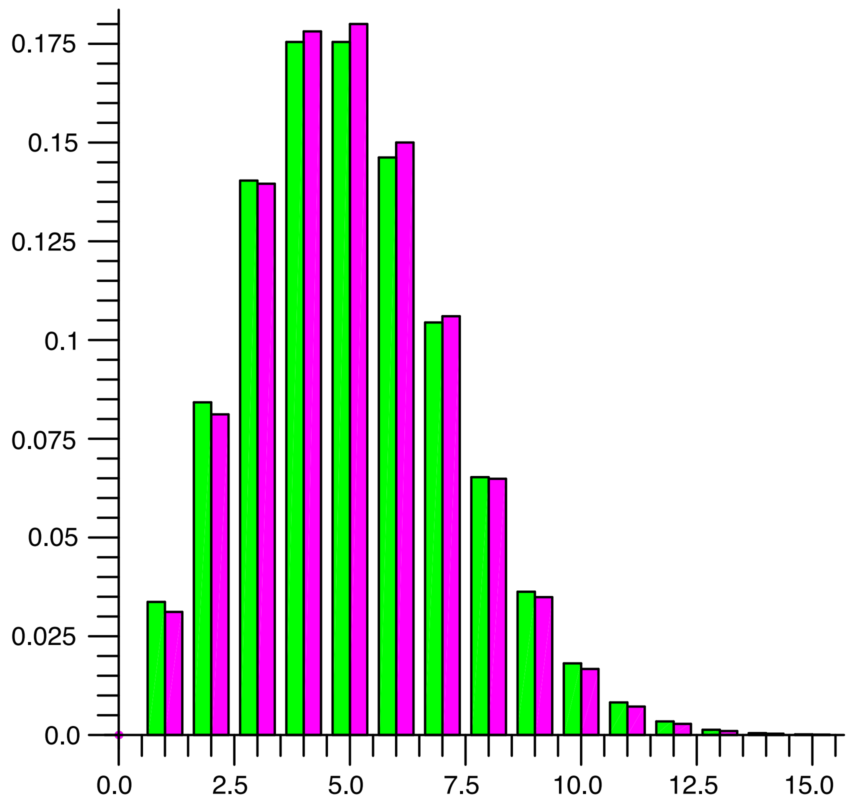


Figure 59: Binomial distribution $B(100, 0.05)$ (dark/magenta) and Poisson distribution with $\lambda = 5$ (light/green). Here $n = 100$ is large, but $np = \lambda$ is fixed.

The Poisson distribution satisfies the normalisation condition (308):

$$\begin{aligned}\sum_{r=0}^{\infty} P(X=r) &= \sum_{r=0}^{\infty} \frac{\lambda^r \exp(-\lambda)}{r!} \\ &= \exp(-\lambda) \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} \\ &= \exp(-\lambda)\exp(\lambda) = 1. \end{aligned} \tag{327}$$

Mean

The mean of the Poisson distribution is

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{r=0}^{\infty} r \frac{\lambda^r \exp(-\lambda)}{r!} \\
&= \exp(-\lambda) \sum_{r=1}^{\infty} \frac{\lambda^r}{(r-1)!} \quad \text{(the } r = 0 \text{ term is zero and can be dropped)} \\
&= \exp(-\lambda)\lambda \sum_{r=1}^{\infty} \frac{\lambda^{r-1}}{(r-1)!} \\
&= \lambda \exp(-\lambda) \sum_{s=0}^{\infty} \frac{\lambda^s}{s!} \quad \text{(writing } r = s + 1\text{)} \\
&= \lambda \exp(-\lambda) \exp(\lambda) \\
&= \lambda \, . \qquad (328)
\end{aligned}$$

So the mean of the Poisson distribution is λ.

Variance

To calculate the variance, it is easiest to find $\mathbb{E}[X^2 - X]$ first:

$$\begin{aligned}
\mathbb{E}[X^2 - X] &= \mathbb{E}[X(X-1)] \\
&= \sum_{r=0}^{\infty} r(r-1)\frac{\lambda^r \exp(-\lambda)}{r!} \\
&= \sum_{r=2}^{\infty} \frac{\lambda^r \exp(-\lambda)}{(r-2)!} \quad \text{(the } r = 0, 1 \text{ terms are zero)} \\
&= \exp(-\lambda)\lambda^2 \sum_{r=2}^{\infty} \frac{\lambda^{r-2}}{(r-2)!} \\
&= \exp(-\lambda)\lambda^2 \sum_{s=0}^{\infty} \frac{\lambda^s}{s!} \quad \text{(writing } r = s + 2\text{)} \\
&= \lambda^2 \quad .
\end{aligned} \tag{329}$$

Hence,

$$\mathbb{E}[X^2] = \mathbb{E}[X^2 - X] + \mathbb{E}[X] = \lambda^2 + \lambda \ , \tag{330}$$

having used (328) for the value of $\mathbb{E}[X]$. Hence, from (316)

$$\sigma^2 = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \lambda \,. \tag{331}$$

7.6 Continuous probability distributions

We now consider a random variable X which can take any value in a continuous range, in general $-\infty < X < \infty$.

Because X is continuous we cannot assign a probability to X taking a single value $X = x$, but instead we can define the probability that X takes a value in an infinitesimally small interval, i.e. $x \leqslant X \leqslant x + \mathrm{d}x$.

We hence define the **probability density function** (PDF) $f(x)$, such that

$$P(x \leqslant X \leqslant x + \mathrm{d}x) = f(x)\,\mathrm{d}x \,. \tag{332}$$

- The probability that X takes a value in the **finite** range $\alpha \leqslant X \leqslant \beta$ is then given by the integral

$$P(\alpha \leqslant X \leqslant \beta) = \int_{\alpha}^{\beta} f(x)\,\mathrm{d}x\ . \tag{333}$$

- The PDF for a continuous random variable is the equivalent of the probability function of a discrete random variable. However, $f(x)$ on its own is *not* a probability.

- We have that $f(x) \geq 0$ to ensure that all probabilities are positive. Also $f(x)$ can be larger than 1 over some subset of possible X, but the integral over any such range must be less than 1.

- The PDF must obey the normalisation condition

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1\ , \tag{334}$$

- The cumulative probability function (CPF), $F(x)$, is defined to be the probability that $X \leqslant x$,

$$F(x) = P(X \leqslant x) = \int_{-\infty}^{x} f(x)\,\mathrm{d}x\ . \tag{335}$$

We can see straightaway that $dF/dx = f(x)$.

7.6.1 Mean and variance

The mean and variance of a continuous random variable can be defined by a simple modification of the definitions for a discrete random variable.

Specifically, the mean is given by

$$\boxed{\mathbb{E}(X) = \mu = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x} \tag{336}$$

and the variance, σ^2, by

$$\boxed{\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\ \mathrm{d}x.} \tag{337}$$

Recall the mean

$$\mathbb{E}[X] \equiv \sum_{i=1}^{n} x_i p_i = \sum_{i=1}^{n} x_i P(X = x_i)$$

and variance

$$\text{var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

of a discrete random variable. The variance is also equal to $\sigma^2 = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$, where

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\ \mathrm{d}x\ . \tag{338}$$

As an example, consider the **uniform distribution** for which the random variable X is uniformly distributed between $X = 0$ and $X = \alpha$, i.e. takes any value in that range equally regularly, where α is a positive constant. This is finite-bandwidth *white noise*, manifesting in, for example, a random number generator on a computer, quantisation error when transferring analogue to digital signals, and is also an approximation to the sound of a crashing cymbal (at least in old school drum machines).

- The PDF is $f(x) = \alpha^{-1}$ for $0 \leq x \leq \alpha$, and 0 otherwise. Obviously $f(x)$ is correctly normalised because $\int_{-\infty}^{\infty} f(x)dx = \int_0^\alpha \alpha^{-1}\, dx = 1$.

- The mean should be $\alpha/2$ by symmetry, but let us check the formula:

$$\mu = \int_0^\alpha x\alpha^{-1} dx = \frac{1}{2}\alpha^{-1}\left[x^2\right]_0^\alpha = \frac{1}{2}\alpha. \tag{339}$$

- What about the variance?

$$\sigma^2 = \langle X^2 \rangle - \mu^2 = \int_0^\alpha x^2 \alpha^{-1} dx - \left(\frac{1}{2}\alpha\right)^2 = \frac{1}{12}\alpha^2. \tag{340}$$

Example 7.9 Find the mean and variance of the exponential distribution

$$f(x) = \begin{cases} \lambda \exp(-\lambda x) & x \geqslant 0 \\ 0 & x < 0\ . \end{cases} \tag{341}$$

What is the probability that X takes a value in excess of two standard deviations from the mean?

We begin by computing the mean μ. Integration by parts helps us here:

$$\begin{aligned} \mu = \int_{-\infty}^{\infty} x f(x)\, dx &= \int_0^\infty \lambda x e^{-\lambda x}\, dx, \\ &= \left[-xe^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x}\, dx, \\ &= 0 + \left[-\frac{1}{\lambda} e^{-\lambda x}\right]_0^\infty = \frac{1}{\lambda}. \end{aligned}$$

Next we compute the variance $\sigma^2 = \mathbb{E}(X^2) - \mathbb{E}(X)^2$. First we work out the first term on the right side:

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^{\infty} \lambda x^2 e^{-\lambda x} dx,$$
$$= \left[-x^2 e^{-\lambda x}\right]_0^{\infty} + 2\int_0^{\infty} x e^{-\lambda x}\,dx$$
$$= 0 + \frac{2}{\lambda}\mu = \frac{2}{\lambda^2}.$$

Therefore, $\sigma^2 = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = 2/\lambda^2 - 1/\lambda^2 = 1/\lambda^2$.

One standard deviation is $\sigma = 1/\lambda$ and the mean is $\mu = 1/\lambda$. We are interested in finding the probability of $|X - \mu| > 2\sigma$, or in other words when $X > 1/\lambda + 2/\lambda = 3/\lambda$ or $X < 1/\lambda - 2/\lambda = -1/\lambda$. From the distribution function we see that there is zero probability of getting a negative value for X, so we discount the second case. The probability of the first case is:

$$P\left(X > \frac{3}{\lambda}\right) = \int_{3/\lambda}^{\infty} \lambda e^{-\lambda x}\,dx = \left[-e^{-\lambda x}\right]_{3/\lambda}^{\infty} = e^{-3} = 0.0498.$$

7.6.2 The normal distribution

The normal (or Gaussian or bell curve) distribution is the most important distribution in statistics. It is ubiquitous in science, as a consequence of the *central limit theorem*: the average of a huge number of random and independent experiments will be distributed increasingly like a normal distribution.

The normal distribution is defined by

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2} \right]} \,. \tag{342}$$

where μ is the mean and σ^2 is variance. It is often denoted $N(\mu, \sigma^2)$.

To handle the normal distribution we need the following pieces of information:

$$\int_{-\infty}^{\infty} \exp(-x^2)\, \mathrm{d}x = \sqrt{\pi} \,, \tag{343}$$

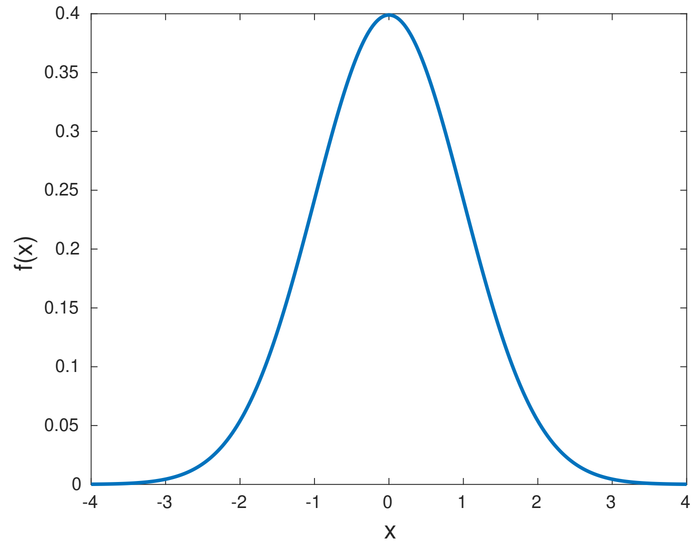


Figure 60: The normal distribution with $\mu = 0$ and $\sigma = 1$.

a result which can be proved using double integrals (see next term);

$$\int_{-\infty}^{\infty} x \exp(-x^2)\, \mathrm{d}x = 0\ , \tag{344}$$

a result which follows straight from the fact that the integrand is an odd function;

and

$$\int_{-\infty}^{\infty} x^2 \exp(-x^2)\ \mathrm{d}x = \frac{\sqrt{\pi}}{2}\ , \tag{345}$$

a result which follows by writing the integrand as $x.x \exp(-x^2)$, integrating by parts and then using (343).

We first check the normalisation condition (334):

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\ \mathrm{d}x &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\ \mathrm{d}x \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-y^2)\sqrt{2}\sigma\ \mathrm{d}y \\ &= 1 \quad . \end{aligned} \tag{346}$$

Mean

The mean follows from

$$\begin{aligned}
\mathbb{E}(X) &= \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\,\mathrm{d}x \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu+\mu) \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\,\mathrm{d}x \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\,\mathrm{d}x \quad \text{since the term with } x-\mu \text{ is odd} \\
&= \mu \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \\
&= \mu \quad \text{using (346)}\,.
\end{aligned} \tag{347}$$

Variance

To find the variance we first compute the expectation value of X^2 as

$$\begin{aligned}
\mathbb{E}(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\, \mathrm{d}x \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \mathrm{d}x \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\mu + \sqrt{2}\sigma y)^2 \exp(-y^2)\, \mathrm{d}y \quad \text{using substitution } x - \mu = \sqrt{2}\sigma y \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\mu^2 + 2\sqrt{2}\sigma\mu y + 2\sigma^2 y^2) \exp(-y^2)\, \mathrm{d}y \\
&= \frac{1}{\sqrt{\pi}} \left[\mu^2\sqrt{\pi} + \sigma^2\sqrt{\pi}\right] \quad \text{using (343, 344 and 345)} \\
&= \mu^2 + \sigma^2 \,,
\end{aligned} \tag{348}$$

which indeed proves that the variance is σ^2.

Cumulative probability distribution

The cumulative probability function for the normal distribution is

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right] \mathrm{d}y \ . \tag{349}$$

The integral here cannot be written in terms of elementary functions. Instead it introduces a new special function, the *error function*, erf(x), defined to be:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2}\, dt,$$

which appears frequently in mathematics and science (especially in problems dealing with diffusion, such as the heat equation). Note that it is an *odd* function.

Hence the cumulative probability distribution can be written as

$$F(x) = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right) \tag{350}$$

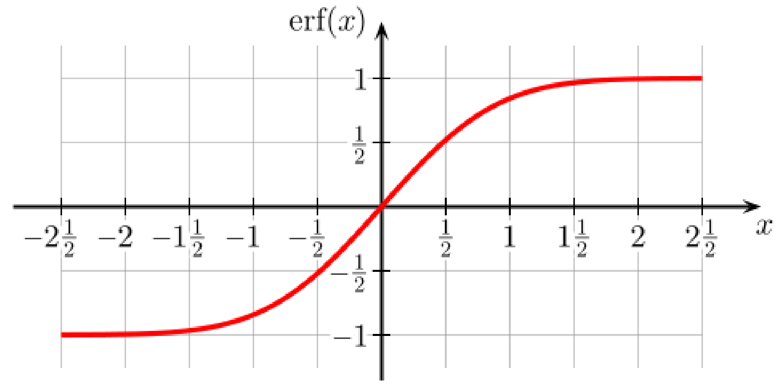


Figure 61: The error function erf(x)

As an example, suppose that a certain manufacturing process produces components whose length is normally distributed with mean 0.5cm and standard deviation 0.005cm, and suppose that a component is rejected if its length differs from the mean by more than 1%.

If we set the random variable X to be the length of the component, then the probability any given component must be rejected is

$$P(\text{reject}) = P(X > 0.5 + 0.005) \;+\; P(X < 0.5 - 0.005)$$
$$= 1 - P(X < 0.505) + P(X < 0.495)$$
$$= 1 - F(0.505) + F(0.495)\ . \qquad (351)$$

We now bring in the error function and get

$$P(\text{reject}) = 1 - \frac{1}{2} - \frac{1}{2}\text{erf}\left(\frac{0.505 - 0.5}{0.005\sqrt{2}}\right) + \frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{0.495 - 0.5}{0.005\sqrt{2}}\right),$$
$$= 1 - \text{erf}\left(\frac{1}{\sqrt{2}}\right),$$

where we have used the oddness of the error function to simplify things.

Computing the error function (using tables or Taylor series), gives us $P(\text{reject}) = 0.3174$. So the proportion of rejected items is a massive 31.74%. Probably best to design a better manufacturing process; one that produces the components with a tighter distribution. (Suppose, in fact, that a new process is implemented and its standard deviation is $\sigma = 0.001$ cm; what is $P(\text{reject})$ now?)

Another example: The Higgs boson detection was often quoted as being at the "5-sigma level". This means the following: if we assume that the Higgs *cannot* exist ever, then the detection at CERN was the result of a random fluctuation that was 5 standard deviations away from the mean.

What is the probability that the fluctuation was actually just noise and not the Higgs, assuming the fluctuations were normally distributed?

We want to work out

$$\begin{aligned} P(|X - \mu| > 5\sigma) &= 1 - F(\mu + 5\sigma) + F(\mu - 5\sigma), \\ &= 1 - \mathrm{erf}(5/\sqrt{2}) \approx 5.733 \times 10^{-7}. \end{aligned}$$

So the probability that the detection was just due to random noise was roughly 0.00006%. It is up to us then to decide whether this is good enough to accept that the Higgs was detected.

- The first gravitational wave detection by LIGO in 2015 of a binary black hole merger (GW150914) was regarded as a 5-sigma result. But a more recent detection of a different colliding black hole binary (GW151012) was only 2-sigma, meaning that the probability that the signal was from something else is roughly 5%. Not nearly as compelling.

- In fact, a result to 2-sigma is regarded in many fields as being sufficiently significant as proof, certainly in drug trials. But it is also means that 1 in 20 published results (using this criterion) are probably wrong . . . .

- And all this presupposes that we have a good model for the random (or other) fluctuations in the system. In other words: is the mean, against which we are basing our sigma, well constrained? A famous example: in 2014 the BICEP2 instrument in Antarctica detected a signal interpreted as evidence of gravitational waves from the primordial universe. Researchers claimed the detection was around 6 sigma. In fact, they had not accounted for the effect of dust in the Milky Way which could easily explain their signal within the probabilities, no outrageous fluctuations needed. The 'mean' was not where they thought it was!

Example 7.10 2007 Paper 1 question 4.

(a) The probability of the number n of persons passing a certain checkpoint during a day is

$$P(n;\lambda) = \frac{\lambda^n \exp(-\lambda)}{n!} ,$$

which defines a Poisson distribution with parameter λ. Show that

$$\sum_{n=0}^{\infty} P(n;\lambda) = 1 .$$

The probability that any given person is male is p. Show that the probability that k males and l females pass the checkpoint during a day is

$$P(k \text{ males, } l \text{ females}) = \binom{k+l}{l} \frac{p^k (1-p)^l \lambda^{(k+l)} \exp(-\lambda)}{(k+l)!} .$$

Hence show that the probability that k males pass (independent of the number of females passing) during the day conforms to a Poisson distribution with parameter λp.

(b) A proportion 0.1 of members of a large population have a certain viral disease, and a further proportion 0.2 are carriers of the virus. A test for the presence of the virus shows positive with probability 0.95 if the person tested has the diseases, 0.9 if the person is a carrier and 0.05 if the person in fact is free of the virus.

Calculate the probability that any given person tests positive.

Calculate the probability that a person who tests negative in fact has the virus (i.e. either has the diseases or is a carrier).

(a) The first part is just asking you to reproduce the notes:

$$\sum_{n=0}^{\infty} P(n;\,\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} = e^{-\lambda} e^{\lambda} = 1,$$

where at the end there we recognised the Taylor series of the exponential function.

In the next part of the question we are interested in the case when $n = k + l$ people pass the checkpoint: k of them are male, and l are female. Putting aside gender, we know that the probability of getting n people past the checkpoint is $P(n;\,\lambda) = \lambda^n e^{-\lambda}/n!$.

Next, we need to multiply this result with the probability that out of these n people, l of them were female. This brings in the binomial distribution, because this is is equivalent to doing n experiments, with each experiment producing one of two outcomes: female or male. We are interested in the probability of getting l 'successes' (i.e. females) in n experiments, given that the probability of getting a single success is $1 - p$. The binomial distribution tells us that the probability of this is

$$^{n}C_l(1-p)^l p^{n-l} = ^{k+l}C_l(1-p)^l p^k.$$

Multiplying the result with our precious probability (and setting $n = k + l$) yields

$$P(k \text{ males}, l \text{ females}) = \frac{\lambda^{k+l} e^{-\lambda}}{(k+l)!} \, ^{k+l}C_l \, p^k (1-p)^l.$$

The final part of (a) wants us to work out the probability that k males pass the checkpoint whatever the number of women who passed. One way to compute this is to work out the probability that k men pass and 0 women pass, then add to that the probability that k men pass and 1 woman passes, then add the probability that k men pass and 2 women pass, and so on and so on. We then have, first rewriting ${}^{k+l}C_l$ in terms of factorials:

$$\begin{aligned} P(k \text{ males}) &= \sum_{l=0}^{\infty} \frac{\lambda^{k+l} e^{-\lambda}}{(k+l)!} {}^{k+l}C_l p^k (1-p)^l, \\ &= \sum_{l=0}^{\infty} \frac{\lambda^{k+l} e^{-\lambda}}{(k+l)!} \frac{(k+l)!}{k!l!} p^k (1-p)^l, \\ &= \frac{\lambda^k e^{-\lambda} p^k}{k!} \sum_{l=0}^{\infty} \frac{\lambda^l (1-p)^l}{l!} \\ &= \frac{(\lambda p)^k e^{-\lambda}}{k!} e^{\lambda(1-p)}, \end{aligned}$$

where in the last line we recognise that $\sum_{l=0}^{\infty} \lambda^l (1-p)^l / l!$ is the Taylor series of $e^{\lambda(1-p)}$ (treating $\lambda(1-p)$ as the x in e^x).

Finally we get

$$P(k \text{ males}) = \frac{(\lambda p)^k e^{-\lambda p}}{k!},$$

which is the Poisson distribution but with $p\lambda$ instead of λ.

(b) To simplify notation, let i indicate 'infected' (i.e. exhibiting the diseases), c indicate 'carrier' (carrying but not exhibiting), and f indicate 'free'. In summary, we then have $P(i) = 0.1$, $P(c) = 0.2$, and $P(f) = 1 - 0.1 - 0.2 = 0.7$.

Next let us denote p as testing positive to the test, and n as testing negative to the test.

The information given us regarding the test can be summarised by:

$$P(p|i) = 0.95, \qquad P(p|c) = 0.9, \qquad P(p|f) = 0.05.$$

And thus $P(n|f) = 1 - 0.05 = 0.95$.

The sets i, c, and f are mutually exclusive, therefore we can write down the total probability of testing positive as:

$$P(p) = P(p \cap i) + P(p \cap c) + P(p \cap f).$$

Using the definition of conditional probability this may be re-expressed as

$$\begin{aligned} P(p) &= P(p|i)P(i) + P(p|c)p(c) + P(p|f)P(f), \\ &= 0.95 \cdot 0.1 + 0.9 \cdot 0.2 + 0.05 \cdot 0.7, \\ &= 0.31 \end{aligned}$$

It also follows that $P(n) = 1 - P(p) = 0.69$.

The last part of the question asks us to find the probability that someone is either a carrier or infected, given that they tested *negative*, in mathematical terms: $P(i \cup c|n)$. To ease notation we write $\overline{f} = i \cup c$. Bayes' law says:

$$P(\overline{f}|n) = \frac{P(n|\overline{f})P(\overline{f})}{P(n)}.$$

We know that $P(\overline{f}) = 0.1 + 0.2 = 0.3$ and $P(n) = 0.69$ but $P(n|\overline{f})$ requires a bit more work. We have the total probability of testing negative as

$$P(n) = P(n \cap f) + P(n \cap \overline{f}) = P(n|f)P(f) + P(n|\overline{f})P(\overline{f}),$$

which gives us

$$P(n|\overline{f})P(\overline{f}) = P(n) - P(n|f)P(f) = 0.69 - 0.95 \cdot 0.7 = 0.025.$$

Therefore

$$P(\overline{f}|n) = \frac{0.025}{0.69} \approx 0.036$$

So about 3.6 %.

Example 7.11 2007 Paper 2 question 3.

(a) The probability of an experiment that involves counting events having the result $N = n$ (where n is a non-negative integer) is

$$P(N = n) = A\rho^n ,$$

where ρ $(0 < \rho < 1)$ is given. Find the normalising constant A. Calculate the probability that $N > n$. Calculate the probability that $N > n$, conditional on $N > m$ (where $n > m$).

(b) The probability density function for a continuous random variable X is

$$f(x) = B\rho^x \equiv B\exp(-\lambda x) \qquad (\lambda = \ln(\rho^{-1}))$$

where x takes values between 0 and ∞. Find the normalising constant B. Calculate the probability that $X > x$, conditional on $X > y$ (assuming $x > y$). Deduce the probability density function for X, conditional on $X > y$. Calculate the variance of X, conditional on $X > y$.

(a) The total probability should be one, i.e. $\sum_{r=0}^{\infty} P(N = r) = 1$. But

$$\sum_{r=0}^{\infty} P(N = r) = A\sum_{r=0}^{\infty} \rho^r$$

is an infinite geometric series that sums to $A/(1 - \rho)$. Hence $A = 1 - \rho$ if the probability distribution is normalised properly.

$P(N > n)$ is just the sum of all probabilities that $N > n$:

$$P(N > n) = \sum_{r=n+1}^{\infty} P(N = r) = \sum_{r=n+1}^{\infty} (1-\rho)\rho^r$$
$$= \sum_{s=0}^{\infty} (1-\rho)\rho^{n+1}\rho^s,$$
$$= (1-\rho)\rho^{n+1}\sum_{s=0}^{\infty}\rho^s,$$
$$= (1-\rho)\rho^{n+1}\frac{1}{1-\rho} = \rho^{n+1}.$$

The last part of this section asks us to find the conditional probability $P(N > n|N > m)$. In other words, given that we have counted m events already, what is the probability that we ultimately count n $(> m)$ or, put another way, another $n - m$ events. From the definition of the conditional probability this can be re-expressed as

$$P(N > n|N > m) = \frac{P((N > n) \cap (N > m))}{P(N > m)} = \frac{P(N > n)}{P(N > m)}$$

because $n > m$ and so $(N > n) \cap (N > m) = N > n$. Using the previous part of the problem, we then have

$$P(N > n|N > m) = \frac{P(N > n)}{P(N > m)} = \frac{\rho^{n+1}}{\rho^{m+1}} = \rho^{n-m}.$$

In other words: the probability of just counting another $n-m$ events more than what we have already. The number of previous events counted does not really enter directly; thus the process has no memory.

(b) First we work out the normalisation from

$$\int_0^\infty f(x)\,dx = B\int_0^\infty e^{-\lambda x}dx = B\left[\frac{-e^{-\lambda x}}{\lambda}\right]_0^\infty,$$

$$= \frac{B}{\lambda} = 1.$$

Thus $B = \lambda$.

The conditional probability is similar to earlier:

$$P(X > x | X > y) = \frac{P((X > x) \cap (X > y))}{P(X > y)} = \frac{P(X > x)}{P(X > y)}$$

the last equality following from $x > y$.

Next we work out $P(X > x)$:

$$P(X > x) = \int_x^\infty \lambda e^{-\lambda z} dz = [-e^{-\lambda z}]_x^\infty = e^{-\lambda x}.$$

Thus we have

$$P(X > x | X > y) = e^{-\lambda(x-y)} = P(X > x - y).$$

This shows that the exponential distribution is 'memoryless', as before. It does not matter how many events we have already counted, the clock 'restarts' (in terms of probability) after each count.

Let us define the new 'conditional' pdf by $g(x)$. Denote its cumulative probability function by $G(x)$. The two are related by $G(x) = \int_0^x g(z)dz$. This definition can be manipulated so that

$$G(x) = 1 - \int_x^\infty g(z)dz = 1 - P(X > x | X > y) = 1 - e^{-\lambda(x-y)}.$$

We then differentiate both sides with respect to x, noting that $dG/dx = g$, which yields $g(x) = \lambda e^{-\lambda(x-y)}$. But note that this is true only for $x > y$. Outcomes corresponding to $x < y$ are impossible (by definition), and thus on this range we set $g = 0$. In summary

$$g(x) = \begin{cases} \lambda e^{-\lambda(x-y)} & x > y \\ 0 & x < y \end{cases}$$

Finally, we compute the variance from the formula

$$\text{Var}(X|X > y) = \mathbb{E}(X^2|X > y) - \mathbb{E}(X|X > y)^2.$$

We first work out the mean:

$$\mathbb{E}(X|X > y) = \int_{-\infty}^{\infty} xg(x)dx = \int_{y}^{\infty} \lambda x e^{-\lambda(x-y)}dx = \int_{0}^{\infty} \lambda(y+z)e^{-\lambda z}dz,$$

where we have made the transformation $z = x - y$. This then gives us

$$\mathbb{E}(X|X > y) = \lambda y \left[\frac{e^{-\lambda z}}{-\lambda}\right]_0^{\infty} + \int_0^{\infty} \lambda z e^{-\lambda z}dz,$$

note that the second term is just the mean of the usual exponential distribution, which we computed earlier in the notes. It is just equal to $1/\lambda$. Hence,

$$\mathbb{E}(X|X > y) = y + \frac{1}{\lambda}.$$

Using integration by parts

$$
\begin{aligned}
\mathbb{E}(X^2|X > y) = \int_y^\infty x^2 \lambda e^{-\lambda(x-y)} dx &= \lambda e^{\lambda y} \left[\frac{-x^2}{\lambda} e^{-\lambda x} \right]_y^\infty + \lambda e^{\lambda y} \int_y^\infty \frac{2x}{\lambda} e^{-\lambda x} dx, \\
&= y^2 + \frac{2}{\lambda} \mathbb{E}(X|X > y), \\
&= y^2 + \frac{2}{\lambda} \left(y + \frac{1}{\lambda} \right).
\end{aligned}
$$

Now putting everything together we have:

$$
\begin{aligned}
\text{Var}(X|X > y) &= \mathbb{E}(X^2|X > y) - \mathbb{E}(X|X > y)^2, \\
&= y^2 + \frac{2}{\lambda} \left(y + \frac{1}{\lambda} \right) - \left(y + \frac{1}{\lambda} \right)^2, \\
&= \frac{1}{\lambda^2}.
\end{aligned}
$$