Natural Sciences Tripos

Part IA Mathematics - Course A

Mathematical Methods I

Examples Sheet 1

Anders C. Hansen

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This sheet provides exercises covering the material contained in the first half of the Michaelmas Term. **ALL** questions should be attempted by students attending Course A lectures.

Basic Skills questions in each section should either be revision of work completed at A-level or provide an opportunity to practise simple examples of the main concepts. These questions can be attempted at anytime and should be relatively quick to complete. Numerical answers to these questions are provided at the back of this sheet.

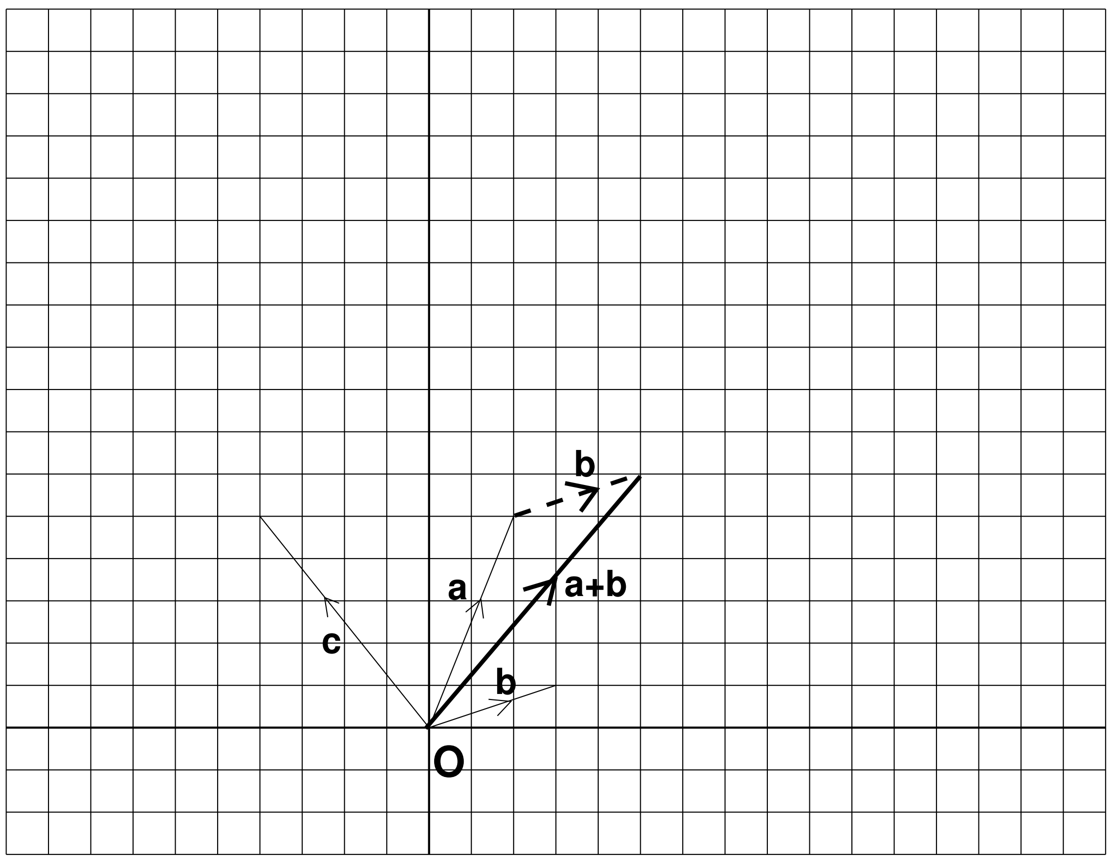
A. Vector Addition and Subtraction

Basic Skills

1. Evaluate the following vector additions and subtractions and represent all vectors in questions (a) - (g) on the graphical diagram below (question (a) has been done as an example).

   Take $\boldsymbol{a} = (2, 5)$; $\boldsymbol{b} = (3, 1)$; $\boldsymbol{c} = (-4, 5)$; $\lambda = 3$; $\mu = 0.5$:

   (a) What is $\boldsymbol{a} + \boldsymbol{b}$?
   (b) What is $\boldsymbol{a} - \boldsymbol{b}$?
   (c) What is $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$?
   (d) What is $\boldsymbol{a} - \boldsymbol{b} - \boldsymbol{c}$?
   (e) What is $\lambda\boldsymbol{a} + \boldsymbol{b}$?
   (f) What is $\boldsymbol{b} - \mu\boldsymbol{a}$?
   (g) What is $\boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$?



   In 3D, take $\boldsymbol{a} = (6, 3, 2)$; $\boldsymbol{b} = (0, 1, 9)$:

   (h) What is $\boldsymbol{a} - \boldsymbol{b}$?

   Take $\boldsymbol{a} = 2\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{k}}$; $\boldsymbol{b} = \hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{k}}$,

   (i) What is $\boldsymbol{a} + \boldsymbol{b}$?

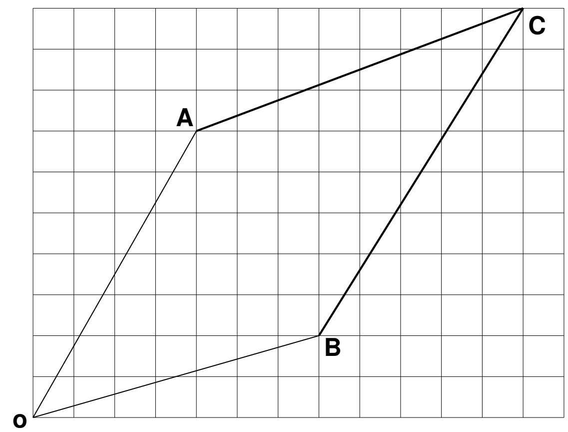
2. If $\overrightarrow{OA} = (2, 3)$ and $\overrightarrow{AB} = (4, 6)$, calculate the vector $\overrightarrow{BO}$.

3. Using the gridded diagram below:

   (a) Work out the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$

   (b) use them to calculate $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$.

   (c) Also show that $\overrightarrow{AB} = -\overrightarrow{BA}$, $\overrightarrow{BC} = -\overrightarrow{CB}$ and $\overrightarrow{AC} = -\overrightarrow{CA}$.



4. Consider 3 points A, B and C with co-ordinates (2,3,1), (6,2,5), and (3,3,8) respectively. Which pair of points are closest together, and what is their relative displacement?

Main Questions

5. If $\overrightarrow{OA} = (1, 2, 3)$ and $\overrightarrow{AC} = (2, -1, -5)$,

   (a) calculate the vector, $\overrightarrow{OC}$.

   (b) Using the cosine rule, calculate the angle between vectors $\overrightarrow{OA}$ and $\overrightarrow{OC}$.

6. An aeroplane has an air speed of 125 km/hr and is flying North. How fast will the plane travel over the Earth, and in what direction, if the wind is:

   (a) From the North at speed 40 km/hr?

   (b) From the South-East at speed 80km/hr?

7. The structure of diamond may be described as a repeating unit cell. The unit cell is a cube with sides of length a, with carbon atoms at each vertex and at the centre of each face (i.e. face-centred cubic). There are additional carbon atoms displaced by $\frac{1}{4}a(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})$ from each of these atoms, where $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$ are the unit vectors along the cube axes.

   (a) Draw the structure of the unit cell

   (b) What are the position vectors of the four nearest neighbours to the atom at $\frac{1}{4}a(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})$?

   (c) What are the vectors joining $\frac{1}{4}a(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})$ to its four nearest neighbours?

8. Two rabbits (Peter and Bugsy) and a fox are positioned in a field as shown in the diagram below. Each square has sides of length 10m. Bugsy spies the fox and begins to run first with a velocity vector (3,4) m/s, Peter starts 10 seconds later with velocity vector (5,2) m/s. The fox starts chasing 10 seconds later than Peter (20 seconds after Bugsy) with velocity vector (5,3) m/s.

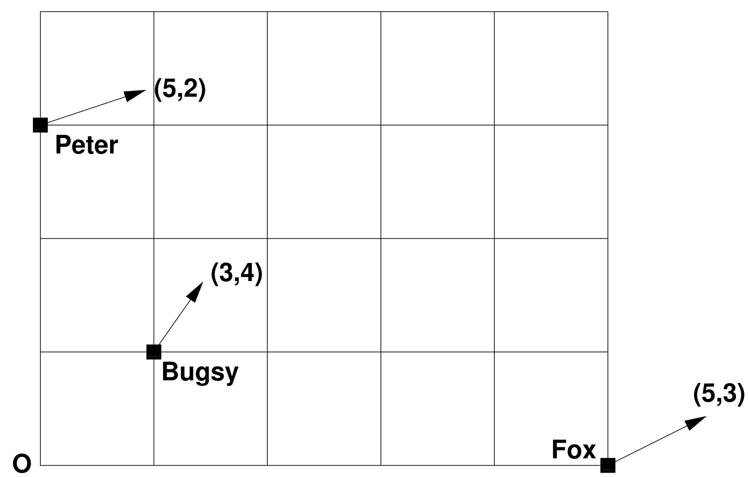
   (a) Does the fox catch a rabbit?

   If so,

   (b) Which one?

   (c) How long does it take the fox?

   (d) How far have each of the animals travelled in that time?



9. ABC is a triangle whose vertices have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$.

   (a) What is the position vector $\boldsymbol{d}$ of the mid-point of BC?

   (b) Write down the position vector $\boldsymbol{p}$ of a point P on the *median* joining A to the mid-point of BC so that it lies a fraction λ along it.

   (c) Write down similar expressions for points on the other two medians.

   (d) By guessing a suitable value of λ (or otherwise) show that the three medians all meet at one point. What is its position vector?

10. (a) Given two position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ write down the equation of any point, $\boldsymbol{r}$, lying on the straight line that passes through $\boldsymbol{a}$ and $\boldsymbol{b}$ in term of $\boldsymbol{a}$, $\boldsymbol{b}$ and a scale factor λ.

    (b) If $\boldsymbol{a} = (1, 1, 0)$ and $\boldsymbol{b} = (3, 1, 4)$, use the equation above to establish which of the following points lie on the straight line passing through $\boldsymbol{a}$ and $\boldsymbol{b}$.

    (i) $\boldsymbol{c} = (-2, 0, -4)$,

    (ii) $\boldsymbol{d} = (5, 1, 8)$,

    (iii) $\boldsymbol{e} = (1, 1, -4)$,

    (iv) $\boldsymbol{f} = (-1/2, 1/2, 2)$.

11. A straight line passes through the points $\boldsymbol{a} = (1, 1, 1)$ and $\boldsymbol{b} = (4, 2, 5)$. A second straight line passes through $\boldsymbol{c} = (3, 1, 6)$ and $\boldsymbol{d} = (2, 0, 7)$.

    Write down their equations,

    (a) using the form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$,

    (b) using the form $\dfrac{x - x_0}{l} = \dfrac{y - y_0}{m} = \dfrac{z - z_0}{n}$.

    Calculate the position vector of the point where the two lines intersect. Show that both equations found in parts (a) and (b) give the same answer.

B. Scalar Product

Basic Skills

1. The vectors $\boldsymbol{a} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}$. Calculate the scalar products $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{b} \cdot \boldsymbol{a}$ directly and verify that they are equal.

2. Suppose $\boldsymbol{c} = (2, 1, 4)$ and $\boldsymbol{d} = (3, -2, 1)$.

    (a) Calculate the angle between the vectors $\boldsymbol{c}$ and $\boldsymbol{d}$.

    (b) Find a vector perpendicular to $\boldsymbol{c}$ and calculate the angle it makes with $\boldsymbol{d}$.

    (c) Find a vector perpendicular to $\boldsymbol{d}$ and calculate the angle it makes with $\boldsymbol{c}$.

3. If $\boldsymbol{e}_1 = (1, 1, 0)$, $\boldsymbol{e}_2 = (1, a, 1)$ and $\boldsymbol{e}_3 = (1, b, -2)$,

    (a) find the values of a and b such that $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ form an orthogonal basis;

    (b) calculate the unit vectors $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_3$ of this basis.

Main Questions

4. The vectors $\boldsymbol{A} = (3, 2, 1)$ and $\boldsymbol{B} = (1, 1, 0)$ and the point $(1, 1, 1)$ lie in a plane.

    (a) Calculate the *unit* normal $\hat{\boldsymbol{n}}$ of the plane.

    (b) Write down the equation of the plane in the form $(\boldsymbol{r} - \boldsymbol{a}) \cdot \hat{\boldsymbol{n}} = 0$.

    (c) Calculate the perpendicular distance of the plane from the origin $(0, 0, 0)$.

5. Consider a plane containing the point with position vector $\boldsymbol{r}$, possessing the unit normal $\hat{\boldsymbol{n}}$, and located a perpendicular distance d from the origin.

    (a) Derive an expression for the shortest distance between the plane and the point P (described by the position vector $\boldsymbol{p}$). Your final result should contain d, $\boldsymbol{p}$, and $\hat{\boldsymbol{n}}$.

    (b) Calculate the shortest distances between the plane $3x - y + 2z = 6$ and the points S, with position vector $(1, -1, 3)$, and T, with position vector $(1, 0, -1)$.

    (c) Do both points S and T lie on the same side of the plane?

6. Consider a plane containing the points with position vectors $\boldsymbol{r}$ and $\boldsymbol{a}$.

   (a) Write down an expression for the unit normal if the the plane is described by $\boldsymbol{b} \cdot (\boldsymbol{r} - \boldsymbol{a}) = 0$?

   A point C lies off the plane with position vector $\boldsymbol{c}$, calculate the shortest distance from the point C to the plane

   (b) Calculate the projection of $\boldsymbol{c}$ onto the plane.

7. Show that the line of intersection of the two planes $x+2y+3z = 0$ and $3x+2y+z = 0$

   (a) is equally inclined to the x and z axes,

   (b) and makes an angle $\cos^{-1}(-\sqrt{2/3})$ with the y-axis.

8. Find the acute angle at which two diagonals of a cube intersect.

9. Four points A, B, C, D are such that AD is perpendicular to BC, and BD is perpendicular to AC. Show that CD is perpendicular to AB.

10. Identify the surfaces:

    (a) $|\mathbf{r}| = d$,

    (b) $\mathbf{r} \cdot \mathbf{u} = e$,

    (c) $\mathbf{r} \cdot \mathbf{u} = f|\mathbf{r}|$,

    (d) $|\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u}| = g$

    where d, e, f and g are fixed scalars and $\boldsymbol{u}$ is a fixed unit vector.

C. Vector Product

Basic Skills

1. Let $\boldsymbol{a} = (2, 1, 3)$, $\boldsymbol{b} = (6, 0, 5)$, and $\boldsymbol{c} = (5, 3, 1)$. Evaluate the following:

   (a) $\boldsymbol{a} \wedge \boldsymbol{b}$ and show that it is perpendicular to both vectors.

   (b) $\boldsymbol{b} \wedge \boldsymbol{a}$ and show that it is perpendicular to both vectors.

   (c) $(\boldsymbol{a} + \boldsymbol{b}) \wedge \boldsymbol{c}$ and show that it is equal to $\boldsymbol{a} \wedge \boldsymbol{c} + \boldsymbol{b} \wedge \boldsymbol{c}$.

   (d) $(\boldsymbol{a} \wedge \boldsymbol{b}) \wedge \boldsymbol{c}$ show that it is not equal to $\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c})$.

   (e) The angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ using the vector product, and check you get the same answer with the scalar product.

   (f) The angle between $\boldsymbol{b}$ and $\boldsymbol{c}$ using the vector product, and check you get the same answer with the scalar product.

(g) The angle between $\boldsymbol{c}$ and $\boldsymbol{a}$ using the vector product, and check you get the same answer with the scalar product.

(h) $\boldsymbol{a} \cdot (\boldsymbol{b} \wedge \boldsymbol{c})$.

(i) $\boldsymbol{c} \cdot (\boldsymbol{a} \wedge \boldsymbol{b})$.

(j) $\boldsymbol{b} \cdot (\boldsymbol{a} \wedge \boldsymbol{c})$.

(k) $\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c})$ and show that it is equal to $(\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$.

Main Questions

2. Find the angle between the position vectors of the points $(2, 1, 1)$ and $(3, -1, -5)$ and find the direction cosines of a vector perpendicular to both.

3. A parallelepiped has one vertex at the origin and the nearest three vertices to the origin at the points A, B, and C, with associated position vectors $(3, 0, 0)$, $(0, 2, 0)$, and $(1, 1, 5)$.

   (a) Calculate the area of the side containing the origin and the points A and B.

   (b) Calculate the volume of the parallelepiped.

4. Show, for any three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, that $(\boldsymbol{b} - \boldsymbol{a}) \wedge (\boldsymbol{c} - \boldsymbol{a}) = \boldsymbol{a} \wedge \boldsymbol{b} + \boldsymbol{b} \wedge \boldsymbol{c} + \boldsymbol{c} \wedge \boldsymbol{a}$.

5. A line is inclined at equal angles to the x, y, z axes and passes through the origin. A second line passes through the points $(1, 2, 4)$ and $(0, 0, 1)$. Find the minimum distance between the two lines using the vector product.

6. You need to drill a straight hole in a piece of metal at right angles to a flat surface containing the points: $(1, 0, 0)$, $(1, 1, 1)$, and $(0, 2, 0)$; and you want the hole to end at the point $(2, 1, 0)$.

   (a) How long a drill must you use?

   (b) Where in the plane given by the flat surface must you start drilling?

7. Suppose $\boldsymbol{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\boldsymbol{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{d} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

   (a) Draw a two-dimensional Cartesian basis set on the plane and mark the points represented by the position vectors $\boldsymbol{c}$ and $\boldsymbol{d}$. Add to this diagram the non-orthogonal basis set represented by $\boldsymbol{a}$ and $\boldsymbol{b}$.

   (b) Write the vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ in terms of the new basis set.

8. Set $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

   Show that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ form a non-orthogonal basis set.

(b) Write the vector $\boldsymbol{d} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ in terms of this basis by using the scalar triple product.

D. Coordinate Systems

Basic Skills

1. By drawing a 2D diagram calculate:
   (a) the x co-ordinate of a point in terms of its plane polar co-ordinates r and θ.
   (b) the y co-ordinate of a point in terms of its plane polar co-ordinates r and θ.

2. By drawing a 2D diagram calculate:
   (a) the r co-ordinate of a point in terms of its Cartesian co-ordinates x and y.
   (b) the θ co-ordinate of a point in terms of its Cartesian co-ordinates x and y.

3. Describe the following loci (given in plane polar co-ordinates),
   (a) $\theta = a$,
   (b) $r = \theta$,

   where a is a constant.

Main Questions

4. By drawing a sphere in Cartesian co-ordinates derive:
   (a) x in terms of r, θ and ϕ,
   (b) y in terms of r, θ and ϕ,
   (c) z in terms of r, θ and ϕ.

5. By drawing a cylinder in Cartesian co-ordinates derive:
   (a) x in terms of r, θ,
   (b) y in terms of r, θ,
   (c) z in terms of z.

6. A point has Cartesian co-ordinates (1,2,5). What are its co-ordinates in:
   (a) cylindrical polar co-ordinates,
   (b) spherical polar co-ordinates.

E. Vector Area

1. Four points define a rectangle in 3D space. Their position vectors are $(2, 0, 7)$, $(5, 0, 3)$, $(5, 4, 3)$, and $(2, 4, 7)$.

   (a) Find the unit normal to the plane of the rectangle, $\hat{\boldsymbol{n}}$.

   (b) What is the total vector area $\boldsymbol{S}$ in terms of $A\hat{\boldsymbol{n}}$?

   (c) Calculate the projected area S_x on to the y-z plane from (b) and verify it graphically

   (d) Calculate the projected area S_y on to the x-z plane from (b) and verify it graphically

   (e) Calculate the projected area S_z on to the x-y plane from (b) and verify it graphically.

2. Points O, B, C, D, and E have position vectors $(0, 0, 0)$, $(2, 0, 0)$, $(2, 2, 0)$, $(0, 2, 0)$ and $(1, 1, 1)$, respectively. What are the vector areas of:

   (a) the square $OBCD$ projected on to the plane with unit normal $\hat{\boldsymbol{n}} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$,

   (b) the upper surface of the pyramid with base $OBCD$ and vertex E,

   (c) a lampshade (truncated hollow cone) whose base is a horizontal circle of radius 4, and whose top is a horizontal circle of radius 3, located a height 5 above the base.

F. Complex Numbers

Basic Skills

1. If $z_1 = 2 + i$ and $z_2 = i - 3$ plot the following numbers on an Argand diagram:

   (a) z_1 (b) z_2
   (c) z_1^* (d) z_2^*
   (e) $z_2 - z_1$ (f) $z_1 - 2z_2$
   (g) iz_1 (h) $z_1 z_2$
   (i) z_1^2 (j) $|z_2|^2$.

2. If $z_1 = 2 + i$ and $z_2 = i - 3$ write the following in the the form $z = |z|e^{i\theta}$

   (a) z_1 (b) z_2
   (c) z_1^* (d) z_2^*
   (e) $z_2 - z_1$ (f) $z_1 - 2z_2$
   (g) iz_1 (h) $z_1 z_2$
   (i) z_1^2 (j) $|z_2|^2$.

3. Find the modulus and principal argument of the following:
   (a) $1 + \sqrt{5}i$
   (b) $-\sqrt{3} - \dfrac{i}{\sqrt{3}}$
   (c) $4i - 3$.

4. If $z = x + iy$, find the real and imaginary parts of the following functions in terms of x and y:
   (a) z^2
   (b) iz
   (c) $(1 + i)z$
   (d) $z^2(z - 1)$.

Main Questions

5. Find the real and imaginary parts of the following complex numbers:
   (a) i^3 (b) i^{4n}
   (c) $\left(\dfrac{i+1}{\sqrt{2}}\right)^2$ (d) $\left(\dfrac{1-i}{\sqrt{2}}\right)^2$
   (e) $\left(\dfrac{1}{2} + \dfrac{\sqrt{3}i}{2}\right)^3$ (f) $\dfrac{1+i}{2-5i}$
   (g) $\left(\dfrac{1+i}{1-i}\right)^2$ (h) $i^{1/3}$
   (i) $(3i - 2)^{1/4}$ (j) i^i.

6. Factorise the following expressions:
   (a) $z^2 + 1$
   (b) $z^2 - 2z + 2$
   (c) $z^2 + i$
   (d) $z^2 + (1 - i)z - i$.

7. Find all the solutions to the following equations, and mark their positions in the Argand diagram.
   (a) $z^3 = -1$
   (b) $z^4 = 1$
   (c) $z^2 = i$
   (d) $z^3 = -i$.

8. Write down the following in the form $a + ib$ where a and b are real.
   (a) $e^{-i\pi/2}$ (b) $e^{-i\pi}$
   (c) $e^{i\pi/4}$ (d) e^{1+i}
   (e) $\exp(2e^{i\pi/4})$ (f) $\exp(re^{i\theta})$, where r and θ are real.

9. Sketch on an Argand diagram the set of points described by the equations:
   (a) $|z| = 4$ (b) $|z-1| = 3$
   (c) $|z-i| = 2$ (d) $|z-(1-2i)| = 3$
   (e) $|z^*-1| = 1$ (f) $|z^*-i| = 1$
   (g) $|z-2| = |z+i|$ (h) $|z-2| = |z^*+i|$
   (i) $|z| = 2|z-2|$ (j) $\arg(z) = \dfrac{\pi}{2}$
   (k) $\arg(z^*) = \dfrac{\pi}{4}$ (l) $\arg(z) = |z|$.

10. By expressing the following in terms of $|z|e^{i\theta}$ find the natural logarithms of:
    (a) $-1 + 0.0001i$ (b) $-1 - 0.0001i$
    (c) i (d) $(1+i)$
    (e) $(x+iy)$ (f) $(x+iy)^*$
    (g) $ire^{i\theta}$.

11. (a) Write down $e^{i\theta}$ in terms of trigonometric functions.

    (b) By considering $e^{in\theta}$ derive *De Moivre's* theorem.

    (c) Use *De Moivre's* theorem to find $\cos 2\theta$ in terms of $\cos\theta$

    (d) Check your result using the trigonometric identity

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

12. Using *De Moivre's* theorem find:

    (a) $\cos 3\theta$ in terms of powers of $\cos\theta$

    (b) $\sin^5\theta$ in terms of $\sin\theta$, $\sin 2\theta$, $\sin 3\theta$, etc.

13. Calculate the following sums:

    (a) $\sum_{n=1}^{5} \sin(n\theta)$

    (b) $\sum_{n=1}^{N} \cos(n\theta)$.

    [Hint: Consider $\sum \exp(in\theta)$.]

14. The position of a particle moving in simple harmonic motion in one dimension is $x = 7 + 24\cos 3t + 7\sin 3t$.

    (a) Show that the displacement from the centre of motion can be written as $\Re(X)$ where $X = (24-7i)e^{3it}$.

    (b) What is the amplitude of the motion?

    (c) Find the time of the first two passages through the centre (after $t = 0$) by writing X in the form $Ae^{i\phi}$ where A and ϕ are real.

    (d) Also find the distance of the stationary points of the particle from the origin.

15. A mass on a spring moves in simple harmonic motion with a frequency, ω. At $t = 0$ it is a distance x_0 from the centre of motion and is moving with a velocity v_0. Find the complex amplitude A in the expression $Ae^{i\omega t}$ that represents the sinusoidal variation in distance of the mass from the centre of the motion.

G. Hyperbolic Functions

1. By writing the trigonometric function in terms of exponential functions (or otherwise) prove the following:

   (a) $\tanh(ix) = i\tan(x)$

   (b) $\text{sech}(ix) = \sec(x)$

   (c) $\cosh^2 x - \sinh^2 x = 1$

   (d) $1 - \tanh^2 x = \text{sech}^2 x$.

2. Find the following trigonometric identities:

   (a) $\cosh(x + y)$ in terms of cosh and sinh functions,

   (b) $\sinh(x + y)$ in terms of cosh and sinh functions,

   (c) $\tanh(x + y)$ in terms of tanh functions.

3. Sketch the graphs of the following functions:

   (a) e^x (b) $\cosh x$

   (c) $\tanh x$ (d) $\log x$

   (e) $\sinh^{-1} x$ (f) $\tanh^{-1} x$.

4. Consider the set of points in the x, y plane described by:

   (a) $(x, y) = (a\cos\theta, b\sin\theta)$,

   (b) $(x, y) = (a\cosh\theta, b\sinh\theta)$,

   where a and b are real and fixed and θ is a real parameter. In each case, find the Cartesian equation of the locus and sketch it.

5. By writing cosh and sinh in terms of exponentials express the following inverse hyperbolic functions in closed form:

   (a) $\sinh^{-1} x$

   (b) $\cosh^{-1} x$

   (c) $\tanh^{-1} x$.

N.B: Further hyperbolic function questions will be included in the calculus section.

Numerical Answers to Basic Skills

A.Vector Addition & Subtraction

1. (a) $(5, 6)$ (b) $(-1, 4)$ (c) $(1, 11)$
   (d) $(3, -1)$ (e) $(9, 16)$ (f) $(2, -1.5)$
   (g) $(9, 10.5)$ (h) $(6, 2, -7)$ (i) $3\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{k}}$

2. $(-6, -9)$

3. (a) $\overrightarrow{OA} = (4, 7)$, $\overrightarrow{OB} = (7, 2)$, $\overrightarrow{OC} = (12, 10)$
   (b) $\overrightarrow{AB} = (3, -5)$ $\overrightarrow{BC} = (5, 8)$, $\overrightarrow{AC} = (8, 3)$
   (c) Show to be true numerically

4. $\overrightarrow{BC} = (-3, 1, 3)$

B. Scalar Product

1. (a) 2
   (b) Show on diagram
   (c) Evaluate numerically and indicate on diagram

2. (a) 62.2^o
   (b) E.g. $(-1, 2, 0)$. Angle $= 146.8^o$.
   (c) E.g. $(0, 1, 2)$. Angle $= 26.6^o$.

3. (a) $a = -1$, $b = -1$
   (b) $\hat{\boldsymbol{e}}_1 = \frac{1}{\sqrt{2}}(1, 1, 0)$, $\hat{\boldsymbol{e}}_2 = \frac{1}{\sqrt{3}}(1, -1, 1)$, $\hat{\boldsymbol{e}}_3 = \frac{1}{\sqrt{6}}(1, -1, -2)$

C. Vector Product

1. (a) $(5, 8, -6)$ (b) $(-5, -8, 6)$ (c) $(-23, 32, 19)$
   (d) $(26, -35, -25)$ (e) 22.5^o (f) 40.8^o
   (g) 43.7^o (h) 43 (i) 43
   (j) -43 (k) $(-39, -81, 53)$

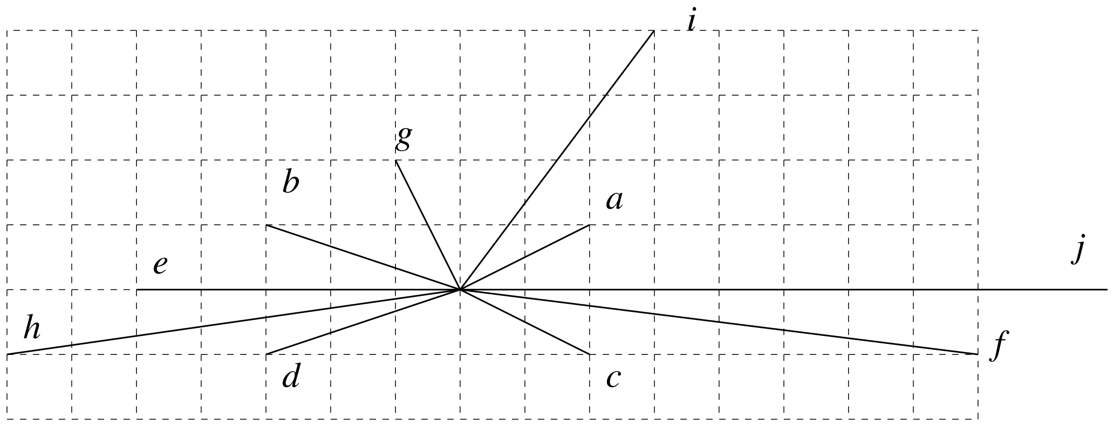
D. Coordinate Systems

1. See lecture notes

2. See lecture notes

3. (a) Straight line at angle a to the$x$ axis.
   (b) Spiral

F. Complex Numbers



1.

2.
(a) modulus = $\sqrt{5}$, arg = 26.6^o
(b) modulus = $\sqrt{10}$, arg = 161.6^o
(c) modulus = $\sqrt{5}$, arg = -26.6^o
(d) modulus = $\sqrt{10}$, arg = -161.6^o
(e) modulus = 5, arg = 180^o
(f) modulus = $\sqrt{65}$, arg = -7.1^o
(g) modulus = $\sqrt{5}$, arg = 116.6^o
(h) modulus = $\sqrt{50}$, arg = -171.9^o
(i) modulus = 5, arg = 53.1^o
(j) modulus = 10, arg = 0^o

3. (a) modulus = $\sqrt{6}$, arg = 65.9^o
(b) modulus = $\sqrt{10/3}$, arg = -161.6^o
(c) modulus = 5, arg = 126.9^o

4. (a) Re = $x^2 - y^2$, Im = $2xy$
(b) Re = $-y$, Im= x
(c) Re = $x - y$, Im= $x + y$
(d) Re = $x^3 - 3xy^2 - x^2 + y^2$, Im = $3x^2y - 2xy - y^3$

Natural Sciences Tripos

Part IA Mathematics - Course A

Mathematical Methods I

Examples Sheet 2

Anders C. Hansen

Michaelmas Term 2023

This sheet provides exercises for the second half of the Michaelmas Term. **ALL** questions should be attempted by students attending Course A lectures. Answers for the basic skills questions are provided at the back of this sheet

H. Calculus Revision

Basic Skills

1. Calculate the following derivatives:

   (a) $\frac{d}{dx}(x^2+3)$ (b) $\frac{d^2}{dx^2}(x^4+2x+6)$

   (c) $\frac{d}{dx}(ax^n+bx+c)$ (d) $\frac{d}{dx}(e^x)$

   (e) $\frac{d}{dt}(at+bt^2\sin\theta)$ (f) $\frac{d}{dx}(1+x^{2/3})$

   (g) $\frac{d}{dy}\left(\frac{1}{1+y}\right)$ (h) $\frac{d}{dx}(\ln(x))$

2. By writing the trigonometric functions in terms of exponential functions evaluate the following derivatives:

   (a) $\frac{d}{dx}(\sin x)$ (b) $\frac{d}{d\theta}(\cos\theta)$

   (c) $\frac{d}{dt}(\tan t)$ (d) $\frac{d}{d\omega}(\sin(-i\omega t))$

   (e) $\frac{d}{dx}(\sinh x)$ (f) $\frac{d}{dx}(\cosh x)$

   (g) $\frac{d}{dx}(\tanh x)$ (h) $\frac{d}{dx}(\tanh(2x))$

3. Calculate the following indefinite integrals:

   (a) $\int x^2 dx$ (b) $\int (ax^n+bx+c)dx$

   (c) $\int e^{2x}dx$ (d) $\int 1/x \ dx$

   (e) $\int \sin(y)dy$ (f) $\int x\cos x dx$

   (g) $\int a\sec^2 x dx$ (h) $\int (2\cos^2 x-1)dx$

4. Calculate the following definite integrals:

   (a) $\int_0^3 (x^2+4)dx$ (b) $\int_0^2 (x-a)^2 dx$

   (c) $\int_0^\pi e^{i\theta}d\theta$ (d) $\int_0^\pi \cos(x)dx$

   (e) $\int_{\pi/4}^{-\pi/4} \text{sech}^2 x dx$ (f) $\int_{-\pi/2}^{\pi/2} \sin(2x)dx$

5. Express the following in terms of partial fractions:

(a) $\dfrac{1}{1-x^2}$

(b) $\dfrac{3x}{2x^2+x-1}$

(c) $\dfrac{2(1-x^2)}{1+x-x^2-x^3}$

(d) $\dfrac{x^4+x^2+4x+6}{3+2x-2x^2-2x^3-x^4}$

Main Questions

6. Calculate the following derivatives:

   (a) $\dfrac{d}{dx}(x\sin x)$ (b) $\dfrac{d}{d\theta}\left(\dfrac{2\theta}{\cos\theta}\right)$

   (c) $\dfrac{d}{dt}(t^2\ln t)$ (d) $\dfrac{d}{dy}(e^y\cos y)$

   (e) $\dfrac{d}{dx}(\cosh x\sinh x)$ (f) $\dfrac{d}{dx}\left(e^{(x^2+2)}\right)$

7. Find $\dfrac{dy}{dx}$ if $y+e^y\sin y=\dfrac{1}{x}$

   (a) by expressing x as a function of y and then finding $\dfrac{dx}{dy}$.

   (b) by differentiating each term with respect to x.

   (c) Show that (a) and (b) give the same answer.

8. By identifying stationary points, asymptotes and intersections sketch the following curves:

   (a) $y=(x-3)^3+2x$ (b) $y=\dfrac{x}{1+x^2}$

   (c) $y=xe^x$ (d) $y=\dfrac{\ln x}{x+1}$

   (e) $y=\dfrac{1}{1-e^x}$ (f) $y=e^x\cos x$.

9. Evaluate the following definite integral
   $$y=\int_1^2\frac{3}{2x^2+x-1}dx.$$

10. Evaluate the following indefinite integrals:

    (a) $\int -\sin x\cos^5 x\ dx$ (b) $\displaystyle\int\frac{\sec^2 x}{\tan x}\ dx$

    (c) $\displaystyle\int\frac{-2}{x^2-1}\ dx$ (d) $\displaystyle\int\frac{3x^2+2}{x^3+2x-1}\ dx$

    (e) $\int x(1+2x)^{-3/2}\ dx$ (f) $\int\sin x\sec x\ dx$

    (g) $\int\cos^4 x\ dx$ (h) $\int\sin^3 x\ dx$

11. With the help of suitable substitutions, find the following indefinite integrals:
    (a) $\displaystyle\int \tan\theta\sqrt{\sec\theta}\; d\theta$
    (b) $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\; dx$
    (c) $\displaystyle\int \frac{6-2x}{(6x-x^2)^{1/2}}\; dx$
    (d) $\displaystyle\int \frac{1}{\sin x}\; dx$
    (e) $\int \sec x \tan x\; dx$

12. Using integration by parts, or otherwise, find the indefinite integrals:
    (a) $\displaystyle\int \ln(x^3)\; dx$
    (b) $\displaystyle\int (\ln x)^3\; dx$

13. Use integration by parts to evaluate the following:
    (a) $\displaystyle\int_0^y x^2 \sin x\; dx$
    (b) $\displaystyle\int_1^y x \ln x\; dx$
    (c) $\displaystyle\int_0^y \sin^{-1} x\; dx$
    (d) $\displaystyle\int_1^y \frac{\ln(a^2+x^2)}{x^2}\; dx$

14. Let $I = \displaystyle\int_0^x e^{at}\cos bt\; dt$ and $J = \displaystyle\int_0^x e^{at}\sin bt\; dt$, where a and b are constants. Use integration by parts to:
    (a) Show that $I = b^{-1}e^{ax}\sin bx - ab^{-1}J$.
    (b) Find another similar relationship between I and J
    (c) Hence find I and J.
    (d) By considering the integral $I + iJ$, find I, J using a different method.

15. Using integration by parts, find a relationship between suitably defined I_n and I_{n-1} or I_{n-2}, where n is any positive integer and hence evaluate the integrals:
    (a) $I_4 = \displaystyle\int_0^1 (1-x^3)^4\; dx$
    (b) $I_6 = \displaystyle\int_0^{\pi/2} x^6 \sin x\; dx$

(c) $I_5 = \int_0^1 x^5 e^x \, dx.$

16. $J(m, n)$, for non-negative integers m and n, is defined by the integral

$$J(m, n) = \int_0^{\pi/2} \cos^m \theta \sin^n \theta \, d\theta$$

(a) Evaluate

(i) $J(0, 0)$ (ii) $J(0, 1)$,
(iii) $J(1, 0)$ (iv) $J(1, 1)$,
(v) $J(m, 1)$ (vi) $J(1, n)$

(b) Using integration by parts, prove that, for m and n both > 1.

$$J(m, n) = \frac{m-1}{m+n} J(m-2, n) \text{ and } J(m, n) = \frac{n-1}{m+n} J(m, n-2),$$

(c) Evaluate

(i) $J(5, 3)$ (ii) $J(6, 5)$ (iii) $J(4, 8)$.

17. State whether the following functions are even, odd or neither:

(a) x (b) $(x-a)^2$
(c) $\sin x$ (d) $\sin(\pi/2 - x)$
(e) $\exp x$ (f) $|x| \cos x$
(g) $\sqrt{x}$ (h) 2
(i) $\ln \left| \frac{(1+x)}{(1-x)} \right|$

I. Power Series

1. (a) Find the first four terms in the Taylor expansion of $\sin x$ about $x = \pi/6$.

(b) Hence find an approximate value for $\sin 31°$.

2. Find the first 2 non-zero terms in the Maclaurin Series of the following.

(a) $\cos x$

(b) $\sin^{-1} x$

(c) e^x

(d) $\ln(x + 1)$

3. (a) Derive an expression for the binomial expansion of $(1 + x)^n$ near $x = 0$ in terms of a sum from 0 to n.

(b) Hence calculate an approximate expression for $(1 + x)^8$ around $x = 0$.

4. Calculate the first three non-zero terms in the binomial approximation for the following:
   (a) $(1+x)^{3/2}$
   (b) $(4+3x)^{1/2}$
   (c) $(3-x)^{-1}$

5. Show that for $|x| \leq 1$

$$\tan^{-1} x = x - x^3/3 + x^5/5 - ...$$

   (a) Using the result $\tan^{-1}(1) = \pi/4$, how many terms of the series are needed to calculate π to 10 decimal places?

   (b) Show that $\pi/4 = \tan^{-1}(1/2) + \tan^{-1}(1/3)$ and deduce another series for π

6. (a) Write down the Taylor series for a function f evaluated at $x+h$ in terms of $f(x)$ and its derivatives evaluated at x. Use this result to show that if x_0 is an approximate solution of the equation $f(x) = 0$, then a better approximation is given, in general, by $x = x_1$ where

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \,.$$

   (b) Sketch the graph of $g(x) = x^3 - 3x^2 + 2$.

   Use the above formula with an initial guess $x_0 = 2.5$ to obtain an improved estimate, x_1 of the largest root of the equation $g(x) = 0$. Apply the method a second time to get a further estimate x_2 (you may leave your answer in rational form).

   Provide a sketch showing the progress of the iterations and demonstrate that the sequence x_0, x_1, x_2, x_3 ... will converge to the root.

J. Probability

1. Two balls are drawn (without replacement) from a box containing 5 blue, 4 green and 1 yellow ball. Like-coloured balls cannot be distinguished.
   (a) Describe the sample space of unordered outcomes.
   (b) Calculate the probability of each outcome.

2. A box of 100 gaskets contains 10 with type A defects only, 5 with type B defects only and 2 with both types of defect.

   Given that a gasket drawn at random has a type A defect find the probability that it also has a type B defect.

3. Show that if there are 23 people in a room, the probability that no two of them share the same birthday is less than 50%.

4. You and a colleague are playing in a gameshow where you both have to select one box each from nine, apparently identical boxes. Six contain a valuable prize but the other three are empty. The host makes you both choose a separate box in turn.

   (a) If you choose first, what is the probability that you win a prize?

   (b) If you choose first and win a prize, what is the probability that your colleague also wins a prize?

   (c) If you choose first and do not win a prize, what is the probability that your colleague does win a prize?

   (d) Is it in your better interest to persuade your colleague to choose first?

5. You randomly choose a biscuit from one of two identical jars. Jar A has 10 chocolate covered biscuits and 30 plain ones. Jar B has 20 chocolate covered and 20 plain biscuits. Unfortunately you choose a plain biscuit. What is the probability that you chose from Jar A?

6. A weighted die gives a probability p of throwing 2,3,4, or 5 , probability $2p$ of throwing 6 and probability $p/2$ of throwing 1.

   (a) Calculate p

   (b) Calculate $\langle x \rangle$, the expected mean score after many throws of the dice.

   (c) In a single throw, what is the probability of obtaining a score higher than $\langle x \rangle$?

   (d) Calculate the variance, σ^2

   (e) Check the formula, $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$.

7. Two duellists, A and B, take alternate shots at each other, and the duel is over when a shot (fatal or otherwise!) hits its target.

   Each shot fired by A has a probability, α, of hitting B

   Each shot fired by B has a probability, β, of hitting A

   (a) Calculate the probability P_1 that A will win if he fires the first shot

   (b) Calculate the probability P_2 that A will win if B fires the first shot

   (c) If they agree to fire at the same time, rather than alternatively, what is the probability P_3 that A will win (i.e. hit B without being hit himself).

8. In the National Lottery, 6 balls are drawn at random from 49 balls, numbered from 1 to 49. You pick 6 different numbers.

   (a) What is the probability that your 6 numbers match those drawn?

   (b) What is the probability that exactly r of the numbers you choose match those drawn?

   (c) What is the probability that 5 numbers you choose match those drawn and your 6th number matches a bonus ball drawn after the other 6?

K. Probability Distributions

1. You arrive home after a big night out and attempt to open your front door with one of the three keys in your pocket. (You may assume that exactly one key will open the door and that if you use it you will be successful).
   Let X, a random variable, be the number of tries that you need to open the door if you take the keys at random from your pocket, but drop any key that fails onto the ground.
   Let Y, another random variable, be the number of tries needed if you take the keys at random from your pocket and immediately put back into your pocket any key that fails.

   Find the probability distribution for X and Y and evaluate $\langle X \rangle$ and $\langle Y \rangle$.

   (*Hint: it may be useful to note that* $1 + 2x + 3x^2 + ... = (1-x)^{-2}$ *if* $|x| < 1$)

2. An opaque bag contains 10 green counters and 20 red. One counter is selected at random and then replaced: green scores 1 and red scores zero. Five draws are made.

   (a) Calculate p_r, the probability of obtaining score $r = \{0, 1, 2, 3, 4, 5\}$. Check that the probabilities sum to 1. Write down the mean $\langle r \rangle$ and variance σ^2 of the score.

   (b) Calculate the probability of obtaining scores in the ranges $\langle r \rangle \pm \sigma/2$, $\langle r \rangle \pm \sigma$.

   (c) The Gaussian approximation of the binomial distribution in (a) is given as: $P_1(r) \propto \exp[-9(r - \frac{5}{3})^2/20]$
   Sketch $P_1(r)$, and p_r.

   (d) Compare your answers in (b) with those for $P_1(r)$. In what sense is $P_1(r)$ a good approximation to p_r?

   (e) Which of your answers would have been different had you not replaced the counters after each selection?

   (f) Which of your answers would have been different had the bag contained only 1 green counter and 2 red counters?

3. The sizes of raindrops in a storm has a probability density function given by:

$$f(s) = \begin{cases} 10ds^2, & 0 \le s \le 0.6, \\ 9d(1-s), & 0.6 < s \le 1, \\ 0, & s > 1, \end{cases}$$

   where d is a constant.

   (a) Find the value of d and sketch the graph of this distribution.

   (b) Write down the most likely size of a raindrop.

   (c) Find the mean size of the raindrops.

   (d) Determine the probability that the size will be: (i) more than 0.8; (ii) between 0.4 and 0.8.

4. The lifetime of a bulb in a traffic signal is a random variable with density:

$$f(t) = \begin{cases} 1, & 1 \le t \le 2, \\ 0, & \text{otherwise,} \end{cases}$$

where t, is measured in years.

(a) What is the probability as a function of y that the failure time of a bulb is less that y years?

(b) The traffic signal contains 3 bulbs. What is the probability as a function of z that none of the bulbs have to be replaced in z years?

5. A certain disease is known to afflict one in a thousand people. You take a medical test that is said to be 99% accurate (i.e. it gives you the correct result in 99% of the cases in which it is used).

   What is the probability that you actually have the disease if the test says you do? Discuss the assumptions implicit in this question and your answer.

6. Suppose that n distinguishable particles are placed randomly into N boxes (states). A particular configuration of this system is such that there are n_s particles in state s, where $1 \le s \le N$. The ordering of particles in any particular state does not matter. Show that the number of ways of realising a particular configuration is:

$$W = n! \prod_{s=1}^{N} \frac{1}{n_s!}$$

[NB: The product symbol $\prod$ is defined such that $\prod_{s=1}^{N} a_s = a_1 a_2 ... a_N$].

Numerical Answers to Basic Skills

H. Calculus Revision

1. (a) $2x$ (b) $12x^2$ (c) $anx^{n-1} + b$
   (d) e^x (e) $a + 2bt\sin\theta$ (f) $\frac{2}{3}x^{-1/3}$
   (g) $\frac{-1}{(1+y)^2}$ (h) $\frac{1}{x}$

2. (a) $\cos x$ (b) $-\sin\theta$ (c) $\sec^2 t$
   (d) $-it\cosh(wt)$ (e) $\cosh x$ (f) $\sinh x$
   (g) $\text{sech}^2 x$ (h) $2\text{sech}^2 2x$

3. (a) $\frac{x^3}{3} + const$ (b) $\frac{a}{n+1}x^{n+1} + \frac{b}{2}x^2 + cx + const$
   (c) $\frac{1}{2}e^{2x} + const$ (d) $\ln x + const$
   (e) $-\cos y + const$ (f) $x\sin x + \cos x + const$
   (g) $a\tan x + const$ (h) $\frac{1}{2}\sin 2x + const$

4. (a) 21 (b) $\frac{8-12a+6a^2}{3}$ (c) 2i
   (d) 0 (e) $-2\tanh(\pi/4) = -1.312$ (f) 0

5. (a) $\frac{1}{2(1-x)} + \frac{1}{2(1+x)}$
   (b) $\frac{1}{2x-1} + \frac{1}{1+x}$
   (c) $\frac{2}{(1+x)}$
   (d) $\frac{2x+3}{(x^2+2x+3)} + \frac{1}{1+x} + \frac{x}{1-x}$