A Priori Regularity of Parabolic Partial Differential Equations

F. Berkemeier and Diogo A. Gomes

Abstract We consider parabolic partial differential equations and develop methods that provide *a priori* estimates for solutions with singular initial data. These estimates are obtained by understanding the time decay of norms of solutions. First, we derive regularity results for the Fokker-Planck equation by estimating the decay of Lebesgue norms. These estimates depend on integral bounds for the advection and diffusion. Then, we apply similar methods to the heat equation. Finally, we conclude by extending our techniques to the porous media equation. The sharpness of our results is confirmed by examining known solutions of these equations. Our main contribution is the use of functional inequalities to establish the decay of norms through nonlinear differential inequalities. These are then combined with ODE methods to deduce estimates for the norms of solutions and their derivatives.

1 Introduction

Parabolic partial differential equations are often used to describe the diffusion of mass, momentum or heat through a material. A classical parabolic PDE is the heat equation:

$$u_t(x,t) = \Delta u(x,t), \tag{1}$$

where $u : \mathbb{R}^d \times [0,T] \to \mathbb{R}$. It is well known that the solution to (1) with singular initial data $u(x,0) = \delta_{x_0}$ is the fundamental solution

$$\Phi(x,t) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}.$$

King Abdullah University of Science and Technology (KAUST), CEMSE Division, Thuwal 23955-6900. Saudi Arabia, D. Gomes e-mail: diogo.gomes@kaust.edu.sa · F. Berkmeier, e-mail: francisco.berkemeier.1@kaust.edu.sa

Although when $t \to 0$, Φ becomes singular, for $t > 0$, Φ is smooth in x and in any L^p space. More precisely, the L^1-norm of this solution is conserved and the L^p-norms decay in time as follows

$$\|\Phi\|_{L^p(\mathbb{R}^d)} = C_p t^{-\frac{1}{2p}d(p-1)}$$

for some constant $C_p > 0$. The preceding identity can be checked by direct computation. Here, we seek to prove similar bounds for solutions of parabolic equations without relying on explicit formulas for the solutions.

We begin by investigating the Fokker-Planck equation

$$u_t(x,t) = \operatorname{div}(b(x,t)u(x,t)) + \operatorname{div}(a(x,t)\nabla u(x,t)),$$

where a is a positive scalar diffusion coefficient and b is a smooth advection vector field. This second-order equation, also known as the Kolmogorov forward equation, models the behavior of a particle under the effect of drag (corresponding to the advection term, b) and random forces (corresponding to the diffusion coefficient, a) and has applications in physics, polymer fluids, plasma, surface physics, and finance, to name just a few. Here, for initial data u_0 and a domain Ω, we obtain estimates of the form

$$\|D^k u\|_{L^p(\Omega)} \leq C\|u_0\|_{L^1(\Omega)}^{f(p,d,k)} t^{-g(p,d,k)},$$

where $k \in \mathbb{N}_0$, $f, g \geq 0$ are functions of the dimension d, k and p, and C is a non-negative constant depending on the space and the problem parameters. Moreover, these estimates depend only on the L^1-norm of the initial data and not on the particular solution.

Our main results on the Fokker-Planck equation are as follows. First, under assumptions on the divergence of the advection, we obtain the theorem:

Theorem 1. *Let u solve (9) with $u \in C^\infty(\mathbb{R}^d \times [0,\infty))$. Let $a > 0$. Moreover, assume $a \in L^{\frac{1}{1-q}}(\mathbb{R}^d)$ for some $1 < q < 2$. Then, for $d \geq 2$, the following holds:*

1. *If $d = 2$ and $1 < q < 2$, or $q < (d+2)/d$ and $d \geq 3$, $\operatorname{div} b = 0$, and $p > 1$, then, for all $t > 0$,*

$$\|u\|_{L^p(\mathbb{R}^d)} \leq C\|u_0\|_{L^1(\mathbb{R}^d)} t^{-\frac{d(p-1)}{p(2-d(q-1))}}. \tag{2}$$

2. *If $\operatorname{div} b \in L^r(\mathbb{R}^d)$ and p, q are such that*

$$2 \leq d < 2r \ \text{ and } \ 1 < q < \frac{2r + dr - d}{dr}, \tag{3}$$

then, there exists $T > 0$ such that

$$\|u\|_{L^p(\mathbb{R}^d)} \leq C\|u_0\|_{L^1(\mathbb{R}^d)} t^{-\frac{d(p-1)}{p(2-d(q-1))}} \tag{4}$$

for all $t < T$. *For* $t > T$, $\|u\|_{L^p(\mathbb{R}^d)} \le C\|u_0\|_{L^1(\mathbb{R}^d)} T^{-\frac{d(p-1)}{p(2-d(q-1))}}$.

Remark 1. The exponent on the right-hand side of (2) is negative if $d = 2$ and $1 < q < 2$, or $q < (d+2)/d$ and $d \ge 3$.

Under integrability assumptions on the advection, we have the following result.

Theorem 2. *Let* u *solve* (9) *with* $u \in C^\infty(\mathbb{R}^d \times [0,\infty))$. *Moreover, assume that* $a^{-1} \in L^r(\mathbb{R}^d)$ *and* $|b| \in L^{\frac{2rq}{r-1}}(\mathbb{R}^d)$ *for some* $q > 1$, $r > 2$. *Then, for any* $p > 1$ *and* $d \ge 2$, *the following holds:*

1. Let q *be such that*

$$q > \frac{d(1-r)}{d-2r} \quad \text{for } 2 < d < 2r. \tag{5}$$

If a *is bounded by above and below, there exists* $T > 0$ *such that*

$$\|u\|_{L^p(\mathbb{R}^d)} \le C\|u_0\|_{L^1(\mathbb{R}^d)} t^{-\frac{qr(p-1)}{p(r+q-1)}} \tag{6}$$

for all $t < T$. *For* $t > T$, $\|u\|_{L^p(\mathbb{R}^d)} \le C\|u_0\|_{L^1(\mathbb{R}^d)} T^{-\frac{qr(p-1)}{p(r+q-1)}}$.
2. Let q *be such that*

$$q > \frac{d(1-r)}{dr(s-1)+d-2r} \quad \text{for } \frac{2}{s} < d < \frac{2r}{1+r(s-1)}. \tag{7}$$

Moreover, if $a \in L^{\frac{1}{1-s}}(\mathbb{R}^d)$, *there also exists* $T > 0$ *such that*

$$\|u\|_{L^p(\mathbb{R}^d)} \le C\|u_0\|_{L^1(\mathbb{R}^d)} t^{-\frac{qr(p-1)}{p(r+q-1)}} \tag{8}$$

for $1 < s < 2$ *and* $t < T$. *For* $t > T$, $\|u\|_{L^p(\mathbb{R}^d)} \le C\|u_0\|_{L^1(\mathbb{R}^d)} T^{-\frac{qr(p-1)}{p(r+q-1)}}$.

The proofs of the prior theorems are presented in Section 2. There, we also discuss an application to L^∞ bounds for the solutions of (9) with singular initial data. Then, in Section 3, we study a particular case, the heat equation. There, we compare our methods with the entropy method [8] and hypercontractivity [1, 5, 7, 11, 12].

Finally, in Section 4, we extend our results to the porous media equation

$$u_t(x,t) = \Delta(u(x,t)^m),$$

where $m \ge 1$. This equation models diffusion processes and fluid flow through porous media (sponge or wood, for example) and has applications in mathematical biology, lubrication, and boundary-layer theory.

Our main contribution is the use of functional inequalities and a differential argument to derive a method to prove estimates for norms of solutions of linear and nonlinear parabolic equations. This method systematizes techniques to infer estimates for solutions of parabolic PDE.

Similar techniques were studied in [2, 14, 15] and used to establish smoothing effects and the time decay of solutions of the heat equation and of the porous media equation. A method comparable to ours was studied in [9, 10]. There, the regularizing effect and the long- and short-time decay were studied for the parabolic Cauchy-Dirichlet problem and the viscous Hamilton-Jacobi equation with a superlinear Hamiltonian.

There are three key techniques used to prove our results. First, we expand the time derivative of the L^p-norms and use integration by parts to establish the decay of these norms. Then, we combine Gagliardo-Nirenberg and Sobolev inequalities with the conservation of L^1-norms to obtain a nonlinear dissipation estimate. Finally, we apply a nonlinear Grönwall-type estimate to get decay in time.

2 Fokker-Planck Equations

Consider the Fokker-Planck equation with initial data in L^1:

$$\begin{cases} u_t(x,t) = \operatorname{div}(b(x)u(x,t)) + \operatorname{div}(a(x)\nabla u(x,t)) & \text{in } \mathbb{R}^d \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^d, \end{cases} \tag{9}$$

where a is a positive scalar diffusion coefficient and b is a smooth advection vector field. In this section, we derive integrability conditions on a and b that imply decay estimates for Lebesgue norms. To simplify the discussion, we assume that a and b are time independent. We are interested in two scenarios. In the first, we assume integrability on the divergence of b. In the second, we assume integrability on b. When $b = 0$ and $a = 1$, (9) becomes the heat equation for which we deduce further regularity in the following section.

2.1 Integrability conditions on the divergence of the advection

Here, we prove Theorem 1 and obtain the two estimates for the solutions of (9) depending on the properties of $\operatorname{div} b$.

Proof (Proof of Theorem 1). 1. We have that

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx = p\int_{\mathbb{R}^d} u^{p-1}\operatorname{div}(bu)\,dx + p\int_{\mathbb{R}^d} u^{p-1}\operatorname{div}(a\nabla u)\,dx. \tag{10}$$

The reverse Hölder inequality, for functions f and g, states that

$$\|fg\|_{L^1(\mathbb{R}^d)} \geq \|f\|_{L^{\frac{1}{q}}(\mathbb{R}^d)} \|g\|_{L^{\frac{1}{1-q}}(\mathbb{R}^d)},$$

whenever $q > 1$. Then, since $a \in L^{\frac{1}{1-q}}(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} u^{p-1} \operatorname{div}(a\nabla u)\, dx = -C \int_{\mathbb{R}^d} a u^{p-2} |\nabla u|^2\, dx \leq -C \left(\int_{\mathbb{R}^d} (u^{p-2}|\nabla u|^2)^{\frac{1}{q}}\, dx \right)^q.$$

Fix $\gamma = p/2$. Then, by the Gagliardo-Nirenberg-Sobolev inequality for $q < 2$, it follows that

$$\left(\int_{\mathbb{R}^d} (u^{p-2}|\nabla u|^2)^{\frac{1}{q}}\, dx \right)^q = \left(\int_{\mathbb{R}^d} |\nabla (u^\gamma)|^{\frac{2}{q}}\, dx \right)^q \geq C \left(\int_{\mathbb{R}^d} u^{\gamma q^*}\, dx \right)^{\frac{2}{q^*}} \quad (11)$$

where q^* is the Sobolev conjugate exponent to $\frac{2}{q}$, given by $q^* = \frac{2d}{dq-2}$. Using the interpolation inequality, L^1-norm conservation, and $0 < \lambda < 1$ with

$$\frac{1}{p} = 1 - \lambda + \frac{\lambda}{\gamma q^*},$$

we have that

$$\left(\int_{\mathbb{R}^d} u^{\gamma q^*}\, dx \right)^{\frac{\lambda}{q^*}} = \|u\|^{\gamma\lambda}_{L^{\gamma q^*}(\mathbb{R}^d)} = \|u\|^{\gamma\lambda}_{L^{\gamma q^*}(\mathbb{R}^d)} \geq \|u\|^{\gamma}_{L^p(\mathbb{R}^d)} \|u_0\|^{\gamma(\lambda-1)}_{L^1(\mathbb{R}^d)},$$

where $\lambda = \frac{d(p-1)}{2+d(p-q)}$. Combining the previous estimates, we get

$$\int_{\mathbb{R}^d} u^{p-1} \operatorname{div}(a\nabla u)\, dx \leq -C \left(\int_{\mathbb{R}^d} u^p\, dx \right)^\beta \|u_0\|^{\frac{2\gamma(\lambda-1)}{\lambda}}_{L^1(\mathbb{R}^d)},$$

where $\beta = \frac{1}{\lambda} = \frac{2+d(p-q)}{d(p-1)}$. For the other term in (10), we have that

$$\begin{aligned} \int_{\mathbb{R}^d} u^{p-1} \operatorname{div}(bu)\, dx &= - \int_{\mathbb{R}^d} u^{p-1} \nabla u \cdot b\, dx \\ &= -C \int_{\mathbb{R}^d} \nabla(u^p) \cdot b\, dx = C \int_{\mathbb{R}^d} u^p \operatorname{div} b\, dx. \end{aligned}$$

Therefore, if $\operatorname{div} b = 0$, with $z(t) = \int_{\mathbb{R}^d} u^p\, dx$, we get the inequality

$$\dot{z} \leq -C \|u_0\|^{\frac{2\gamma(\lambda-1)}{\lambda}}_{L^1(\mathbb{R}^d)} z^\beta.$$

Thus, by Lemma 1,

$$z(t) \leq C \|u_0\|^{\frac{2\gamma(\lambda-1)}{\lambda(1-\beta)}}_{L^1(\mathbb{R}^d)} t^{\frac{1}{1-\beta}} = C \|u_0\|^p_{L^1(\mathbb{R}^d)} t^{-\frac{d(p-1)}{2-d(q-1)}},$$

which yields the estimate in (2).

2. Now, we assume that $\operatorname{div} b \in L^r(\mathbb{R}^d)$. Hence, Hölder's inequality leads to

$$\int_{\mathbb{R}^d} u^p \operatorname{div} b\, dx \leq \left(\int_{\mathbb{R}^d} u^{pr'}\, dx\right)^{\frac{1}{r'}} \left(\int_{\mathbb{R}^d} (\operatorname{div} b)^r\, dx\right)^{\frac{1}{r}},$$

where $1/r' + 1/r = 1$. From (11), we have

$$\int_{\mathbb{R}^d} u^{p-1} \operatorname{div}(a\nabla u)\, dx \leq -C \left(\int_{\mathbb{R}^d} u^{\gamma q^*}\, dx\right)^{\frac{2}{q^*}},$$

where $\gamma = p/2$ and $q^* = 2d/(dq-2)$. Then, we have that

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^p\, dx \leq C\left(\int_{\mathbb{R}^d} u^{pr'}\, dx\right)^{\frac{1}{r'}} - C\left(\int_{\mathbb{R}^d} u^{\gamma q^*}\, dx\right)^{\frac{2}{q^*}}. \tag{12}$$

Note that, by interpolation,

$$\left(\int_{\mathbb{R}^d} u^{pr'}\, dx\right)^{\frac{1}{r'}} \leq \left(\int_{\mathbb{R}^d} u^{\gamma q^*}\, dx\right)^{\frac{p\theta}{\gamma q^*}} \|u_0\|_{L^1(\mathbb{R}^d)}^{p(1-\theta)},$$

where θ is such that $\frac{1}{pr'} = \frac{\theta}{\gamma q^*} + 1 - \theta$. Note that the previous inequality only holds if

$$pr' < \gamma q^*;$$

that is

$$\frac{pr}{r-1} < \frac{pd}{dq-2},$$

which is true if (3) holds. Therefore, with $y(t) = \int_{\mathbb{R}^d} u^{\gamma q^*}\, dx$, we have that the right-hand side of (12) is bounded by

$$C_1\|u_0\|_{L^1(\mathbb{R}^d)}^{p(1-\theta)} y^{\frac{p\theta}{\gamma q^*}} - C_2 y^{\frac{2}{q^*}} = C_1\|u_0\|_{L^1(\mathbb{R}^d)}^{p(1-\theta)} y^{\frac{2\theta}{q^*}} - C_2 y^{\frac{2}{q^*}}.$$

Then, since $\theta < 1$, with $z(t) = \int_{\mathbb{R}^d} u^p\, dx$, we have that, using Lemma 2 and interpolation again, there exists $T > 0$ such that, for all $t < T$,

$$\begin{aligned}\dot z \leq -Cy^{\frac{2}{q^*}} &= -C\left(\int_{\mathbb{R}^d} u^{\gamma q^*}\, dx\right)^{\frac{2}{q^*}} \leq -C\left(\int_{\mathbb{R}^d} u^p\, dx\right)^{\frac{1}{\lambda}} \|u_0\|_{L^1(\mathbb{R}^d)}^{\frac{2\gamma(\lambda-1)}{\lambda}} \\ &= -C\|u_0\|_{L^1(\mathbb{R}^d)}^{\frac{2\gamma(\lambda-1)}{\lambda}} z^{\frac{1}{\lambda}},\end{aligned}$$

where $\lambda = \frac{d(p-1)}{2+dp-dq}$. Then, we get

$$z(t) \leq C\|u_0\|_{L^1(\mathbb{R}^d)}^{\frac{2\gamma(\lambda-1)}{\lambda(1-1/\lambda)}} t^{\frac{1}{1-1/\lambda}} = C\|u_0\|_{L^1(\mathbb{R}^d)}^{p} t^{\frac{d(p-1)}{d(q-1)-2}},$$

and thus (4) follows, for all $t < T$. For $t > T$,

$$\|u\|_{L^p(\mathbb{R}^d)} \leq C\|u_0\|_{L^1(\mathbb{R}^d)} T^{-\frac{d(p-1)}{p(2-d(q-1))}}.$$

■

2.2 Integrability conditions on the advection

We now recall Theorem 2, where we consider integrability on the advection.

Proof (Proof of Theorem 2). We have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx &= C\int_{\mathbb{R}^d} u^{p-1}\operatorname{div}(bu)\,dx + C\int_{\mathbb{R}^d} u^{p-1}\operatorname{div}(a\nabla u)\,dx\\ &= -C\int_{\mathbb{R}^d} u^{p-1}\nabla u\cdot b\,dx - C\int_{\mathbb{R}^d} au^{p-2}|\nabla u|^2\,dx\\ &= -C\int_{\mathbb{R}^d} a^{\frac{1}{2}}u^{\frac{p}{2}-1}\nabla u\cdot bu^{\frac{p}{2}}a^{-\frac{1}{2}}\,dx - C\int_{\mathbb{R}^d} au^{p-2}|\nabla u|^2\,dx.\end{aligned}$$

Then, reorganizing the previous inequality and using Cauchy's inequality with ϵ, we have that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx + C\int_{\mathbb{R}^d} au^{p-2}|\nabla u|^2\,dx &= -C\int_{\mathbb{R}^d} a^{\frac{1}{2}}u^{\frac{p}{2}-1}\nabla u\cdot bu^{\frac{p}{2}}a^{-\frac{1}{2}}\,dx\\ &\leq \left|C\int_{\mathbb{R}^d} a^{\frac{1}{2}}u^{\frac{p}{2}-1}\nabla u\cdot bu^{\frac{p}{2}}a^{-\frac{1}{2}}\,dx\right| \qquad (13)\\ &\leq \epsilon C\int_{\mathbb{R}^d} |a|u^{p-2}|\nabla u|^2\,dx + C_\epsilon\int_{\mathbb{R}^d} |b|^2u^p|a|^{-1}\,dx.\end{aligned}$$

Hence, for ϵ small, we can rewrite (13) as

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx + C\int_{\mathbb{R}^d} au^{p-2}|\nabla u|^2\,dx \leq C_\epsilon\int_{\mathbb{R}^d} |b|^2u^p|a|^{-1}\,dx. \qquad (14)$$

Now, applying Hölder's inequality twice to the last term in the previous inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^d} |b|^2u^p|a|^{-1}\,dx &\leq \left(\int_{\mathbb{R}^d} |b|^{2r'}u^{pr'}\right)^{\frac{1}{r'}}\left(\int_{\mathbb{R}^d} |a|^{-r}\right)^{\frac{1}{r}}\\ &\leq C\left(\int_{\mathbb{R}^d} u^{pr'q'}\right)^{\frac{1}{r'q'}}\left(\int_{\mathbb{R}^d} |b|^{2r'q}\right)^{\frac{1}{r'q}} \leq C\left(\int_{\mathbb{R}^d} u^{pr'q'}\right)^{\frac{1}{r'q'}},\end{aligned}$$

where $\frac{1}{r}+\frac{1}{r'}=1=\frac{1}{q}+\frac{1}{q'}$ and $r'q=\frac{rq}{r-1}$. Accordingly, defining $\gamma=pr'q'=\frac{pqr}{(q-1)(r-1)}$, we have, from (14),

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx \le C_1\left(\int_{\mathbb{R}^d} u^\gamma\,dx\right)^{\frac{p}{\gamma}} - C_2\int_{\mathbb{R}^d} a|\nabla(u^{\frac{p}{2}})|^2\,dx,$$

where $C_1, C_2>0$ are constants depending on η and ϵ. Now, we consider the two cases separately.

1. If a is bounded by above and below, then, by Sobolev's inequality, we have that

$$\int_{\mathbb{R}^d} a|\nabla(u^{\frac{p}{2}})|^2\,dx \ge C\left(\int_{\mathbb{R}^d} u^{\frac{2^*p}{2}}\,dx\right)^{\frac{2}{2^*}}.$$

Then, using interpolation, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx &\le C_1\left(\int_{\mathbb{R}^d} u^\gamma\,dx\right)^{\frac{p}{\gamma}} - C_2\left(\int_{\mathbb{R}^d} u^{\frac{2^*p}{2}}\,dx\right)^{\frac{2}{2^*}}\\ &\le C_1\left(\int_{\mathbb{R}^d} u^{\frac{2^*p}{2}}\,dx\right)^{\frac{2\theta}{2^*}}\|u_0\|_{L^1(\mathbb{R}^d)}^{p(1-\theta)} - C_2\left(\int_{\mathbb{R}^d} u^{\frac{2^*p}{2}}\,dx\right)^{\frac{2}{2^*}},\end{aligned}$$

where θ is such that $\frac{1}{\gamma}=\frac{2\theta}{2^*p}+1-\theta$. Note that the previous inequality only holds if $\gamma\le 2^*p/2$. This is true for q such that (5) holds. Hence, since $\theta<1$, using Lemma 2 and interpolation again, there exists $T>0$ such that, for all $t<T$,

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx \le -C\left(\int_{\mathbb{R}^d} u^p\,dx\right)^{\frac{1}{\lambda}}\|u_0\|_{L^1(\mathbb{R}^d)}^{\frac{p(\lambda-1)}{\lambda}}$$

for some $\lambda>0$ such that $\frac{1}{p}=\frac{\lambda}{\gamma}+1-\lambda \Leftrightarrow \lambda=\frac{\gamma(p-1)}{p(\gamma-1)}$, which yields

$$\lambda=\frac{qr(p-1)}{qr(p-1)+q+r-1}.$$

Hence, setting $z(t)=\int_{\mathbb{R}^d} u^p\,dx$, we get an inequality of the type $\dot z\le -C\|u_0\|_{L^1(\mathbb{R}^d)}^{\frac{p(\lambda-1)}{\lambda}} z^{\frac{1}{\lambda}}$. Thus,

$$z(t)\le C\|u_0\|_{L^1(\mathbb{R}^d)}^p t^{\frac{1}{1-1/\lambda}} = C\|u_0\|_{L^1(\mathbb{R}^d)}^p t^{\frac{qr(1-p)}{r+q-1}},$$

which combined with (5), yields (6), for $t<T$. For $t>T$, $\|u\|_{L^p(\mathbb{R}^d)}\le C\|u_0\|_{L^1(\mathbb{R}^d)}T^{-\frac{qr(p-1)}{p(r+q-1)}}$.

2. If $a\in L^{\frac{1}{1-s}}(\mathbb{R}^d)$, by Hölder's reverse inequality, we have that

$$\int_{\mathbb{R}^d} a|\nabla(u^{\frac{p}{2}})|^2\,dx \ge \left(\int_{\mathbb{R}^d} a^{\frac{1}{1-s}}\,dx\right)^{1-s}\left(\int_{\mathbb{R}^d}|\nabla(u^{\frac{p}{2}})|^{\frac{2}{s}}\,dx\right)^s$$

$$\geq C\left(\int_{\mathbb{R}^d} |\nabla(u^{\frac{p}{2}})|^{\frac{2}{s}}\, dx\right)^s.$$

Then, for $s < 2$, the Gagliardo-Nirenberg-Sobolev inequality yields

$$\left(\int_{\mathbb{R}^d} |\nabla(u^{\frac{p}{2}})|^{\frac{2}{s}}\, dx\right)^s \geq C\left(\int_{\mathbb{R}^d} u^{\frac{mp}{2}}\, dx\right)^{\frac{2}{m}}$$

with $m = \frac{2d}{ds-2}$. Furthermore, interpolation and L^1-norm conservation yield

$$\left(\int_{\mathbb{R}^d} u^\gamma\, dx\right)^{\frac{p}{\gamma}} \leq \left(\int_{\mathbb{R}^d} u^{\frac{mp}{2}}\, dx\right)^{\frac{2\theta}{m}} \|u_0\|_{L^1(\mathbb{R}^d)}^{p(1-\theta)}, \tag{15}$$

where θ is such that $\frac{1}{\gamma} = \frac{2\theta}{mp} + 1 - \theta$. Note that (15) holds if $\gamma < mp/2$. This is true for q such that (7) holds. Then, following the same steps as before, since $\theta < 1$, we have that there exists $T > 0$ such that, for all $t < T$,

$$z(t) \leq C\|u_0\|_{L^1(\mathbb{R}^d)}^p t^{\frac{1}{1-1/\lambda}} = C\|u_0\|_{L^1(\mathbb{R}^d)}^p t^{\frac{qr(1-p)}{r+q-1}},$$

which combined with (7), yields (8), for $1 < s < 2$ and $t < T$. For $t > T$, $\|u\|_{L^p(\mathbb{R}^d)} \leq C\|u_0\|_{L^1(\mathbb{R}^d)} T^{-\frac{qr(p-1)}{p(r+q-1)}}$. ■

2.3 The adjoint method

One application of our estimates are bounds of the form

$$\|v(\cdot,0)\|_{L^\infty(\Omega)} \leq C\|f\|_{L^b([0,T],L^q(\Omega))} \tag{16}$$

for solutions of

$$\begin{cases} v_t + b\cdot\nabla v = \operatorname{div}(a\nabla v) + f & \text{in } \Omega\times(0,T] \\ v(x,T) = v_T(x) & \text{in } \Omega, \end{cases} \tag{17}$$

where $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$ and $v_T \in W^{1,\infty}(\Omega)$. To prove those bounds, we use the adjoint method. Estimates such as (16) arise in the theory of mean-field games, for example. As in [3, 4, 6], the adjoint problem to (17) is

$$\begin{cases} u_t = \operatorname{div}(ub) + \operatorname{div}(a\nabla u) & \text{in } \Omega\times(0,T] \\ u(x,0) = \delta_{x_0} & \text{in } \Omega. \end{cases} \tag{18}$$

The central idea in the adjoint method is to derive a representation formula for solutions of (17) in terms of solutions of (18). Arguing as in [6], we have

$$v(\cdot,0)=\int_0^T\int_\Omega f(x,t)u(x,t)\,dxdt+\int_\Omega v_T(x)u(x,T)\,dx.$$

Then, it follows that

$$|v(\cdot,0)|\le\int_0^T\int_\Omega |f(x,t)u(x,t)|\,dxdt+\int_\Omega |v_T(x)u(x,T)|\,dx. \tag{19}$$

Therefore, to estimate the left-hand side, it is enough to bound each of the two terms on the right-hand side of the prior inequality. For the second term on the right-hand side, we have that, by Hölder's inequality,

$$\int_\Omega |v_T(x)u(x,T)|\,dx\le\|v_T\|_{L^\infty(\Omega)}\|u(x,T)\|_{L^1(\Omega)}=\|v_T\|_{L^\infty(\Omega)}\le C,$$

since $v_T\in W^{1,\infty}(\Omega)$. For the first term in (19), we apply Hölder's inequality twice to conclude that

$$\begin{aligned}\int_0^T\int_\Omega |fu|\,dxdt&\le\int_0^T\|f\|_{L^q(\Omega)}\|u\|_{L^p(\Omega)}\,dt\\&\le\|f\|_{L^b([0,T],L^q(\Omega))}\|u\|_{L^c([0,T],L^p(\Omega))},\end{aligned}\tag{20}$$

where $\frac{1}{b}+\frac{1}{c}=1=\frac{1}{p}+\frac{1}{q}$. Thus, we see that by getting bounds for u, we can convert them into bounds for v. Therefore, the estimates from Theorems 1 and 2, which still hold for the Fokker-Planck equation with singular initial data, yield estimates for $\|v(\cdot,0)\|_{L^\infty(\Omega)}$. We have the following result.

Theorem 3. *Let v,u solve* (17) *and* (18)*, respectively, in $\mathbb{R}^d$. Let $\frac{1}{b}+\frac{1}{c}=1=\frac{1}{p}+\frac{1}{q}$. Then,*

1. *Under the assumptions of Theorem 1, if*

$$c>\frac{p(2-d(q-1))}{d(p-1)}, \tag{21}$$

then $\|v(\cdot,0)\|_{L^\infty(\mathbb{R}^d)}\le C\|f\|_{L^b([0,T],L^q(\mathbb{R}^d))}$.

2. *Under the assumptions of Theorem 2, if*

$$c>\frac{p(r+q-1)}{qr(p-1)},$$

then $\|v(\cdot,0)\|_{L^\infty(\mathbb{R}^d)}\le C\|f\|_{L^b([0,T],L^q(\mathbb{R}^d))}$.

Proof. 1. By (20), we have that

$$\|v(\cdot,0)\|_{L^\infty(\mathbb{R}^d)}\le\|f\|_{L^b([0,T],L^q(\mathbb{R}^d))}\|u\|_{L^c([0,T],L^p(\mathbb{R}^d))}.$$

Then, by Theorem 1,

$$\|u\|^c_{L^c([0,T],L^p(\mathbb{R}^d))} = \int_0^T \|u\|^c_{L^p(\mathbb{R}^d)}\,dt \leq C\int_0^T t^{-\frac{cd(p-1)}{p(2-d(q-1))}}\,dt,$$

which is finite if and only if (21) holds. Hence, the estimate follows.

2. The proof is analogous using Theorem 2. ■

3 The Heat Equation

Here, we apply the methods from the previous section to the homogeneous heat equation, which corresponds to (9) for $b = 0$ and $a = 1$:

$$\begin{cases} u_t(x,t) = \Delta u(x,t) & \text{in } \Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{22}$$

We consider the cases where $\Omega = \mathbb{R}^d$ and $\Omega = \mathbb{T}^d$.

3.1 Main estimate

We now give an estimate for the L^p-norm of a derivative of any order of the solution of the heat equation.

Theorem 4. *Let u solve* (22) *with $u \in C^\infty(\Omega \times [0,\infty))$. Then, there exists $T > 0$ such that, for any $k \in \mathbb{N}_0$, $p > 1$, the following estimate holds*

$$\|D^k u\|_{L^p(\Omega)} \leq C\|u_0\|_{L^1(\Omega)} t^{-\frac{dp+kp-d}{2p}} \tag{23}$$

for all $t > 0$ with $\Omega = \mathbb{R}^d$ and for $t \in [0,T)$ with $\Omega = \mathbb{T}^d$. For $t > T$, the norm is bounded.

Proof. Fix $\gamma = p/2$. Then,

$$\frac{d}{dt}\int_\Omega |D^k u|^p\,dx = C\int_\Omega |D^k u|^{p-2} D^k u D^k \Delta u\,dx = -C\int_\Omega |\nabla(|D^k u|^\gamma)|^2\,dx.$$

For $\Omega = \mathbb{R}^d$, by Sobolev and Gagliardo-Nirenberg inequalities, we have that

$$C\left(\int_{\mathbb{R}^d} |\nabla(|D^k u|^\gamma)|^2\,dx\right)^{\frac{\lambda}{2}} \geq \|D^k u\|^{\gamma\lambda}_{L^{2^*\gamma}(\mathbb{R}^d)} \geq \|D^k u\|^\gamma_{L^p(\mathbb{R}^d)} \|u_0\|^{\gamma(\lambda-1)}_{L^1(\mathbb{R}^d)},$$

where $\lambda = \frac{d(p-1)+kp}{2+d(p-1)+kp}$ satisfies

$$\frac{1}{p} = 1 - \lambda + \frac{k}{d} + \lambda\left(\frac{1}{2^*\gamma} - \frac{k}{d}\right).$$

Then, with $z(t) = \int_{\mathbb{R}^d} |D^k u|^p \, dx$, we get the inequality

$$\dot{z} \leq -C\|u_0\|_{L^1(\mathbb{R}^d)}^{\frac{2\gamma(\lambda-1)}{\lambda}} z^{\frac{1}{\lambda}}.$$

Thus, by Lemma 1, $z(t) \leq C\|u_0\|_{L^1(\mathbb{R}^d)}^p t^{\frac{1}{1-1/\lambda}}$ and (23) follows since

$$\frac{1}{1/\lambda - 1} = \frac{1}{2}(d(p-1) + kp).$$

For $\Omega = \mathbb{T}^d$, the Gagliardo-Nirenberg inequality for bounded domains yields

$$\left(\int_{\mathbb{T}^d} |D^k u|^{2^*\gamma} \, dx \right)^{\frac{1}{2^*}} \leq C \left(\int_{\mathbb{T}^d} |D^k u|^{2\gamma} \, dx + \int_{\mathbb{T}^d} |\nabla(|D^k u|^\gamma)|^2 \, dx \right)^{\frac{\alpha}{2}}.$$

Next, we observe that by Gagliardo-Nirenberg inequality

$$\left(\int_{\mathbb{T}^d} |D^k u|^p \, dx \right)^{\frac{1}{p}} \leq C \left(\int_{\mathbb{T}^d} |D^k u|^{2^*\gamma} \, dx \right)^{\frac{\lambda}{2^*}} \|u_0\|_{L^1(\mathbb{T}^d)}$$

where

$$\frac{1}{p} - \frac{k}{d} = 1 - \lambda + \lambda \left(\frac{1}{2^*\gamma} - \frac{k}{d} \right).$$

The preceding identity yields

$$\lambda = \frac{d(p-1) + kp}{2 + d(p-1) + kp}.$$

Then, fixing $z(t) = \int_{\mathbb{T}^d} |D^k u|^p \, dx$, we get the following differential inequality

$$\dot{z} \leq C_1 z - C_2 \|u_0\|_{L^1(\mathbb{T}^d)}^{\gamma \frac{\lambda - 1}{\lambda}} z^{\frac{\gamma}{\lambda p}}$$

Hence, by Lemma 2, there exists $T > 0$ such that z satisfies

$$z(t) \leq C\|u_0\|_{L^1(\mathbb{T}^d)}^p t^{\frac{1}{1-1/\lambda}} = C\|u_0\|_{L^1(\mathbb{T}^d)}^p t^{-\frac{1}{2}(d(p-1)+kp)}$$

for $t \in [0, T)$. Thus, we get a similar estimate for $\mathbb{T}^d$. Also, by the same lemma, the norm is bounded for $t > T$. ■

Remark 2. Comparing again with the fundamental solution, we have that

$$\int_{\mathbb{R}^d} |D^k \Phi|^p \, dx \leq C t^{-\frac{dp}{2} - \frac{kp}{2}} \int_{\mathbb{R}^d} e^{-C\frac{p|x|^2}{t}} \, dx = C t^{-\frac{dp + kp - d}{2}},$$

which is the same estimate as (23). Hence, our estimates are as sharp as possible.

In the following two sections, we compare our method with two alternative approaches: the entropy and hypercontractivity methods.

3.2 Entropy methods

Now, we follow the discussion in [8] for the Fokker-Planck equation and present the entropy method applied to the heat equation. We define the entropy

$$H(t) = \int_{\mathbb{R}^d} \phi(u)\, dx,$$

where u solves (22) and ϕ is a convex function. Integration by parts yields

$$\dot{H}(t) = \frac{d}{dt}\int_{\mathbb{R}^d} \phi(u)\, dx = -\int_{\mathbb{R}^d} \phi''(u)|\nabla u|^2\, dx \leq 0.$$

Furthermore,

$$\ddot{H}(t) = -\int_{\mathbb{R}^d} \phi^{(3)}(u)u_t|\nabla u|^2 + 2\phi''(u)\nabla u\cdot\nabla(u_t)\, dx = I_1 + I_2,$$

where

$$I_1 = -\int_{\mathbb{R}^d} \phi^{(3)}(u)u_t|\nabla u|^2\, dx = \int_{\mathbb{R}^d} \phi^{(4)}(u)|\nabla u|^4 + 2\phi^{(3)}(u)\Delta u|\nabla u|^2\, dx$$

and

$$I_2 = -2\int_{\mathbb{R}^d} \phi''(u)\nabla u\cdot\nabla(u_t)\, dx = 2\int_{\mathbb{R}^d} \phi^{(3)}(u)\Delta u|\nabla u|^2 + \phi''(u)(\Delta u)^2\, dx.$$

Hence,

$$\ddot{H}(t) = \int_{\mathbb{R}^d} \phi^{(4)}(u)|\nabla u|^4 + 4\phi^{(3)}(u)\Delta u|\nabla u|^2 + 2\phi''(u)(\Delta u)^2\, dx.$$

We now set $\phi(u) = u^2$. Accordingly,

$$\ddot{H}(t) = 4\int_{\mathbb{R}^d} (\Delta u)^2\, dx.$$

Hence, for some constant, $C > 0$, the Gagliardo-Nirenberg inequality yields

$$\ddot{H}(t) = 4\int_{\mathbb{R}^d} (\Delta u)^2\, dx \geq C\left(\int_{\mathbb{R}^d} |\nabla u|^2\, dx\right)^{\alpha} = C(-\dot{H}(t))^{\alpha},$$

where α satisfies $\frac{1}{2} = \frac{2}{d} + \left(\frac{1}{2} - \frac{1}{d}\right)\alpha + 1 - \alpha$. Hence,

$$z(t) = -\dot{H}(t)$$

satisfies the following differential inequality

$$\dot{z} \leq -Cz^{\alpha}.$$

Hence, as before, $\dot{H}$ satisfies

$$|\dot{H}(t)| \leq Ct^{\frac{1}{1-\alpha}}$$

and thus, for some C depending on α,

$$\int_{\mathbb{R}^d} u^2\,dx = H(t) \leq Ct^{1+\frac{1}{1-\alpha}} = Ct^{-\frac{d}{2}},$$

which is the same estimate as the one obtained from Theorem 4 with $k = 0$ and $p = 2$. We have then shown that our technique gives similar results to entropy methods.

3.3 On logarithmic Sobolev inequalities and hypercontractivity

The gain of regularity in time can also be understood using the results in [2, 7, 11] on logarithmic Sobolev inequalities and hypercontractivity. Contractivity principles, which appear in quantum field theory, are often used to describe operators as contractions between Lebesgue spaces, being of particular interest the case from L^p to L^q when $p \leq q$.

Next, we state a result from [5] that yields a generalization of the logarithmic Sobolev inequality presented in [7]. First, we recall that the Fenchel-Legendre transform of a convex function φ is the function $\varphi^* : \mathbb{R}^d \to \mathbb{R}$ given by

$$\varphi^*(\mu) = \sup_{x \in \mathbb{R}^d} \{\mu \cdot x - \varphi(x)\}.$$

Proposition 1 (Gentil-Gross). *Let φ be a C^1 strictly convex function on $\mathbb{R}^d$ such that*

$$\lim_{|x| \to +\infty} \frac{\varphi(x)}{\|x\|} = +\infty.$$

Then, for all $\lambda > 0$ and for any smooth function g on $\mathbb{R}^d$, we have the following Euclidean logarithmic Sobolev inequality

$$\int_{\mathbb{R}^d} e^g \log\left(\frac{e^g}{\int_{\mathbb{R}^d} e^g\,dx}\right) dx \leq -d\log(\lambda e)\int_{\mathbb{R}^d} e^g\,dx + \int_{\mathbb{R}^d} \varphi^*(-\lambda\nabla g)e^g\,dx. \tag{24}$$

We begin by considering a time-dependent Lebesgue norm. More specifically, we are interested in bounding

$$\|u\|_{L^{s(t)}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} u^{s(t)}\,dx\right)^{\frac{1}{s(t)}},$$

where $1 \leq s(t) < \infty$.

Proposition 2. *Let u be a solution to the d-dimensional heat equation* (22)*. Assume that $1 \leq s(t) < \infty$ is a nondecreasing function, with $s(0) = p \geq 1$ and such that*

$$s(t) = 1 + (p-1)e^{\frac{2t}{\lambda^2}}, \tag{25}$$

where $\lambda = e^{-1}$. Then, the following estimate holds

$$\|u\|_{L^{s(t)}(\mathbb{R}^d)} \leq \|u_0\|_{L^p(\mathbb{R}^d)} \tag{26}$$

for all $t \geq 0$.

Proof. Let $s \equiv s(t)$. As before, we have that

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^s\,dx = s\int_{\mathbb{R}^d} u^{s-1}\Delta u\,dx + \dot{s}\int_{\mathbb{R}^d} u^s \log u\,dx. \tag{27}$$

We have that

$$\begin{aligned} s\int_{\mathbb{R}^d} u^{s-1}\Delta u\,dx &= -s(s-1)\int_{\mathbb{R}^d} u^{s-2}|\nabla u|^2\,dx \\ &= -\frac{4(s-1)}{s}\int_{\mathbb{R}^d} |\nabla(u^{\frac{s}{2}})|^2\,dx \leq 0. \end{aligned} \tag{28}$$

Fix $g = \log(u^s)$ in (24) to get

$$\int_{\mathbb{R}^d} u^s \log\left(\frac{u^s}{\int_{\mathbb{R}^d} u^s\,dx}\right) dx \leq -d\log(\lambda e)\int_{\mathbb{R}^d} u^s\,dx + \int_{\mathbb{R}^d} \varphi^*(-\lambda\nabla\log(u^s))u^s\,dx.$$

Taking $\lambda = e^{-1}$, we estimate the second term on the right-hand side of (27) as

$$\dot{s}\int_{\mathbb{R}^d} u^s\log u\,dx \leq \frac{\dot{s}}{s}\left[\int_{\mathbb{R}^d} \varphi^*(-\lambda\nabla\log(u^s))u^s\,dx + \log\left(\int_{\mathbb{R}^d} u^s\,dx\right)\int_{\mathbb{R}^d} u^s\,dx\right]. \tag{29}$$

Fix $\varphi(x) = \frac{|x|^2}{2}$. Then, $\varphi^*(\mu) = \frac{|\mu|^2}{2}$. Thus

$$\int_{\mathbb{R}^d} \varphi^*(-\lambda \nabla \log(u^s)) u^s \, dx = \frac{1}{2}(\lambda s)^2 \int_{\mathbb{R}^d} |\nabla u|^2 u^{s-2} \, dx = 2\lambda^2 \int_{\mathbb{R}^d} |\nabla (u^{\frac{s}{2}})|^2 \, dx. \tag{30}$$

Then, combining (27), (28), (29) and (30), we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^d} u^s \, dx \leq (g(t) - f(t)) \int_{\mathbb{R}^d} |\nabla (u^{\frac{s}{2}})|^2 \, dx + \frac{\dot{s}}{s} \log \left(\int_{\mathbb{R}^d} u^s \, dx \right) \int_{\mathbb{R}^d} u^s \, dx,$$

where

$$f(t) = \frac{4(s(t)-1)}{s(t)} \quad \text{and} \quad g(t) = \frac{2\lambda^2 \dot{s}(t)}{s(t)}.$$

Now, we select $\dot{s} \geq 0$ such that

$$g - f = 0;$$

that is,

$$\dot{s} = \frac{2s-2}{\lambda^2}, \tag{31}$$

whose solution is (25). Hence, for s such that (31) holds, we have the following differential inequality

$$\frac{d}{dt} \int_{\mathbb{R}^d} u^s \, dx \leq \frac{\dot{s}}{s} \log \left(\int_{\mathbb{R}^d} u^s \, dx \right) \int_{\mathbb{R}^d} u^s \, dx.$$

Fix $z(t) = \int_{\mathbb{R}^d} u^s \, dx$ and $h(t) = \frac{\dot{s}}{s} = \frac{d}{dt} \log(s(t))$. Thus, the previous inequality simplifies to

$$\dot{z}(t) \leq h(t) \log(z(t)) z(t).$$

Rewriting, we get

$$\frac{\dot{z}(t)}{\log(z(t)) z(t)} \leq h(t)$$

and thus

$$\frac{d}{dt}(\log(\log(z(t)))) \leq \frac{d}{dt} \log(s(t)).$$

Finally, with $s(0) = p$, integrating the prior expression leads to

$$\log(\log(z(t))) \leq \log(s(t)) + \log(\log(z(0))) - \log p$$

and thus

$$z(t) \leq \exp\{\exp\{\log(s(t)) + \log(\log(z(0))) - \log p\}\} = z(0)^{\frac{s(t)}{p}} = \|u_0\|_{L^p(\mathbb{R}^d)}^{s(t)}.$$

Hence, (26) follows. ■

Remark 3. 1. If $s(0)=1$ in the previous proposition, (25) forces $s(t)=1$ for all $t \geq 0$, which makes (26) trivial.

2. For t such that $s(t)>2$, interpolation yields, for some $\lambda(t)$,

$$\|\Phi\|_{L^2(\mathbb{R}^d)} \leq \|\Phi\|_{L^1(\mathbb{R}^d)}^{\lambda(t)} \|\Phi\|_{L^{s(t)}(\mathbb{R}^d)}^{1-\lambda(t)} \leq C.$$

By the estimate in Theorem 4, we have $\|\Phi\|_{L^2(\mathbb{R}^d)} \leq C t^{-\frac{d}{4}}$. Hence, our estimate still yields a sharper result.

3. For the fundamental solution Φ of the heat equation, the prior hypercontractivity result yields $\|\Phi\|_{L^{s(t)}(\mathbb{R}^d)} \leq C$, where C is a fixed constant. On the other hand, a direct estimate yields

$$\|\Phi\|_{L^{s(t)}(\mathbb{R}^d)} = (4\pi t)^{-\frac{d}{2}} \left(\int_{\mathbb{R}^d} e^{-\frac{s(t)|x|^2}{4t}} \, dx \right)^{\frac{1}{s(t)}} = s(t)^{-\frac{d}{2s(t)}} (4\pi t)^{-\frac{d(s(t)-1)}{2s(t)}}.$$

Since $s(t)^{-\frac{d}{2s(t)}} \to 1$ and $\frac{s(t)-1}{s(t)} \to 1$ as $t \to \infty$, we have that the hypercontractivity estimate does not provide information about the decay of Lebesgue norms.

3.3.1 Estimate curves

We are now interested in finding a norm function, $s(t)$, for a specific estimate. We start by analyzing estimates for the fundamental solution. By Remark 3, for a fixed estimate $a>0$, the curve $s(t)$ such that $\|\Phi\|_{L^{s(t)}(\mathbb{R}^d)} = a$ is given implicitly by

$$s(t) = a^{-\frac{2s(t)}{d}} (4\pi t)^{1-s(t)}. \tag{32}$$

Using a numerical solver in *Mathematica*, with $d=3$, Figure 1 shows the curve $s(t)$ for different time intervals and values of a.

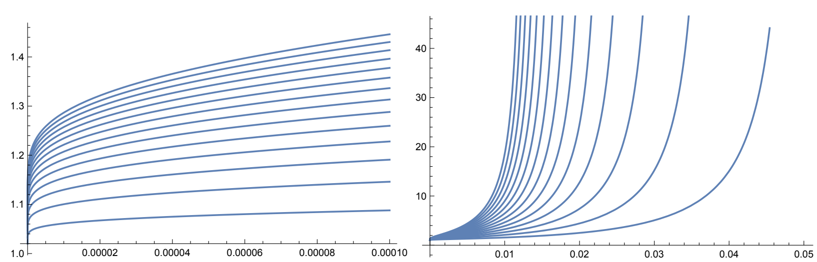

Fig. 1 $s(t)$ paths for $2 \leq a \leq 15$ up to $t=0.0001$ and $t=0.05$

Here, we are considering solutions of (32) such that $s(t) \geq 1$. Such solutions only occur up to a certain time T_a, depending on a, which defines a vertical asymptote of $s(t)$. Using *Mathematica* again, we conclude that, for any dimension, T_a is given explicitly by $T_a = 1/(4\pi a^{\frac{2}{d}})$. Next, we deduce a similar estimate for the curves $\tilde{s}(t)$ regarding the result from Theorem 4 for general solutions of (22). Fixing $s(t) = p$ and $\gamma = p/2$, we have that, by Sobolev's inequality and interpolation,

$$\begin{aligned} \frac{d}{dt}\|u\|^p_{L^p(\mathbb{R}^d)} &= -\frac{4(p-1)}{p}\int_{\mathbb{R}^d} |D(u^\gamma)|^2\,dx \\ &\leq -\frac{4(p-1)}{pC_d^2}\left(\int_{\mathbb{R}^d} u^{2^*\gamma}\,dx\right)^{\frac{2}{2^*}} \\ &\leq -\frac{4(p-1)}{pC_d^2}\|u\|_{L^1(\mathbb{R}^d)}^{\frac{2\gamma(\lambda-1)}{\lambda}}\left(\int_{\mathbb{R}^d} u^p\,dx\right)^{\frac{1}{\lambda}}, \end{aligned}$$

where $\lambda = \frac{d(p-1)}{2+d(p-1)}$ and C_d is the Sobolev's inequality constant, which only depends on the dimension. By [13], the sharp Sobolev's constant is given explicitly by

$$C_d = (\pi d(d-2))^{-\frac{1}{2}}\left(\frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)}\right)^{\frac{1}{d}}.$$

Then, similar to the proof of Theorem 4, we have that

$$\|u\|_{L^p(\mathbb{R}^d)} \leq \left(\frac{4(p-1)(1/\lambda-1)}{pC_d^2}\right)^{\frac{1}{p(1-1/\lambda)}}\|u\|_{L^1(\mathbb{R}^d)}t^{\frac{1}{p(1-1/\lambda)}}.$$

Now, for a fixed a, the curve $\tilde{s}(t)$ such that $\|u\|_{L^{\tilde{s}(t)}(\mathbb{R}^d)} \leq a$ is given implicitly by

$$2^{-\frac{3d(\tilde{s}(t)-1)}{2\tilde{s}(t)}}\left(\frac{1}{\tilde{s}(t)}(d-2)\pi^{1+\frac{1}{d}}\left(2^{d-1}\Gamma\left(\frac{d+1}{2}\right)\right)^{-\frac{2}{d}}\right)^{-\frac{d(\tilde{s}(t)-1)}{2\tilde{s}(t)}}\|u\|_{L^1(\mathbb{R}^d)}t^{-\frac{d(\tilde{s}(t)-1)}{2\tilde{s}(t)}} = a.$$

With $d = 3$ and $\|u\|_{L^1(\mathbb{R}^d)} = 1$, Figure 2 shows the curve $\tilde{s}(t)$ for different time intervals and values of a.

We now compare both norm curves. Fix a such that $\|\Phi\|_{L^{s(t)}(\mathbb{R}^d)} = a$. Figure 3 illustrates $\|\Phi\|_{L^{\tilde{s}(t)}(\mathbb{R}^d)}$, for different values of a.

Hence, for all $t > 0$, $\|\Phi\|_{L^{\tilde{s}(t)}} \leq a$ and norm decay is still verified. Furthermore, we also compare the nature of both norms near $t = 0$, by studying the limit of $\frac{s(t)-1}{\tilde{s}(t)-1}$ as $t \to 0$. Figure 4 suggests that $\lim_{t\to 0}\frac{s(t)-1}{\tilde{s}(t)-1} < \infty$, also indicating that $s(t)$ and $\tilde{s}(t)$ might have similar behavior near $t = 0$.

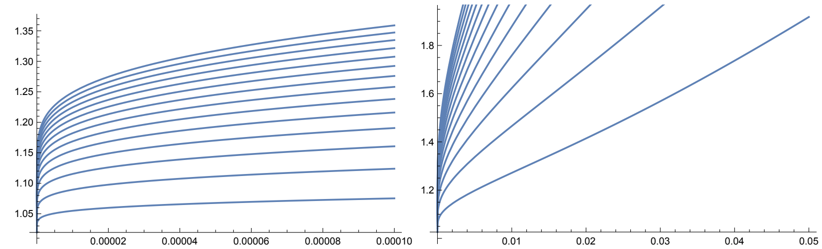

Fig. 2 $\tilde{s}(t)$ paths for $2 \leq a \leq 15$ up to $t = 0.0001$ and $t = 0.05$

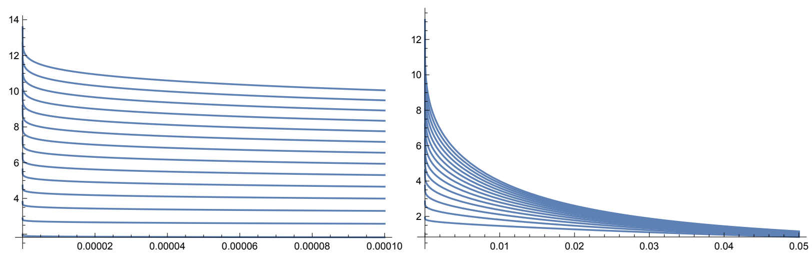

Fig. 3 $\|\Phi\|_{L^{\tilde{s}(t)}(\mathbb{R}^d)}$ for $2 \leq a \leq 15$ up to $t = 0.0001$ and $t = 0.05$

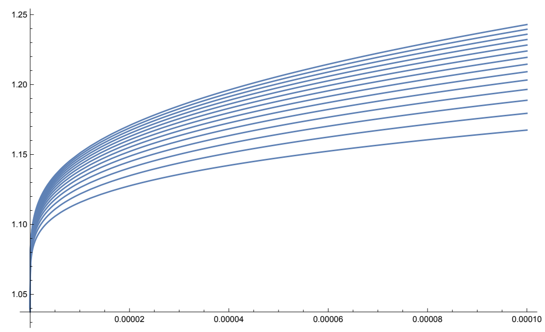

Fig. 4 $\frac{s(t)-1}{\tilde{s}(t)-1}$

4 The Porous Media Equation

The porous media equation (PME) is the following PDE

$$\begin{cases} u_t(x,t) = \Delta(u(x,t)^m) & \text{in } \mathbb{R}^d \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^d, \end{cases} \tag{33}$$

for some $m \in [1, \infty)$ and where we take $u \geq 0$. Note that $m = 1$ corresponds to the heat equation. Here, we extend the ideas from the previous sections to obtain integrability estimates for the solution of the PME. Next, we examine the Barenblatt solutions to show that our bounds are sharp. We conclude this section by comparing our method with the results in [15].

4.1 Estimate methods revisited

We begin by applying our method to (33).

Theorem 5. *Let u solve* (33) *with $u \in C^\infty(\mathbb{R}^d \times [0,\infty))$. Then, for $p \geq 1$, the following estimate holds*

$$\|u\|_{L^p(\mathbb{R}^d)} \leq C\|u_0\|_{L^1(\mathbb{R}^d)}^{\frac{d(m-1)+2p}{p(d(m-1)+2)}} t^{-\frac{d(p-1)}{p(d(m-1)+2)}} \tag{34}$$

for all $t > 0$.

Proof. We begin by noticing that

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx = p\int_{\mathbb{R}^d} u^{p-1}\Delta(u^m)\,dx = -mp(p-1)\int_{\mathbb{R}^d} u^{m+p-3}|\nabla u|^2\,dx \leq 0. \tag{35}$$

Fix $\gamma = (m+p-1)/2$. Then, (35) yields

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx = -C\int_{\mathbb{R}^d} u^{2\gamma-2}|\nabla u|^2\,dx = -C\int_{\mathbb{R}^d} |\nabla(u^\gamma)|^2\,dx. \tag{36}$$

By Sobolev inequality, we have that

$$\left(\int_{\mathbb{R}^d} u^{2^*\gamma}\,dx\right)^{\frac{1}{2^*}} \leq C\left(\int_{\mathbb{R}^d} |\nabla(u^\gamma)|^2\,dx\right)^{\frac{1}{2}}. \tag{37}$$

Using the interpolation inequality and $0 < \lambda < 1$, we have that

$$\begin{aligned}\left(\int_{\mathbb{R}^d} u^{2^*\gamma}\,dx\right)^{\frac{2\lambda}{2^*}} &= \|u\|_{L^{2^*\gamma}(\mathbb{R}^d)}^{2\gamma\lambda} = \|u\|_{L^{2^*\gamma}(\mathbb{R}^d)}^{2\gamma\lambda} \\ &\geq \|u\|_{L^p(\mathbb{R}^d)}^{2\gamma}\|u_0\|_{L^1(\mathbb{R}^d)}^{2\gamma(\lambda-1)},\end{aligned} \tag{38}$$

where $\lambda = \frac{d(p-1)(m+p-1)}{p(2+d(m+p-2))}$. Hence, (36), (37) and (38) lead to

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx \leq -C\|u_0\|_{L^1(\mathbb{R}^d)}^{\frac{2\gamma(\lambda-1)}{\lambda}}\left(\int_{\mathbb{R}^d} u^p\,dx\right)^{\frac{2\gamma}{\lambda p}}.$$

Let $z(t) = \int_{\mathbb{R}^d} u^p\,dx$. Then, the previous inequality can be written as $\dot{z} \leq -C\|u_0\|_{L^1(\mathbb{R}^d)}^{\frac{2\gamma(\lambda-1)}{\lambda}} z^\beta$, where $\beta = 2\gamma/(\lambda p) > 1$. As before, we get the following time estimate

$$z(t) \le C\|u_0\|_{L^1(\mathbb{R}^d)}^{\frac{2\gamma(\lambda-1)}{\lambda(1-\beta)}} t^{\frac{1}{1-\beta}} = C\|u_0\|_{L^1(\mathbb{R}^d)}^{\frac{d(m-1)+2p}{p(d(m-1)+2)}} t^{\frac{d(1-p)}{d(m-1)+2}}$$

and, thus, (34) follows. ■

Next, we consider an estimate for a known solution to (33) and compare it to the prior estimate.

4.2 Barenblatt solutions

The Barenblatt solution of the PME has the following explicit formula, for an arbitrary constant $C > 0$,

$$\mathcal{U}(x,t) = t^{-\alpha}(C - k|x|^2 t^{-2\sigma})_+^{\frac{1}{m-1}},$$

where $(s)_+ = \max\{s, 0\}$ and

$$\alpha = \frac{d}{d(m-1)+2}, \quad \sigma = \frac{\alpha}{d}, \quad k = \frac{\alpha(m-1)}{2md}.$$

Denote the ball centered at the origin with radius $R = (Ct^{2\sigma}/k)^{\frac{1}{2}}$ by B_R. Then, with $u = \mathcal{U}$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{U}^p\,dx &= \int_{B_R} \mathcal{U}^p\,dx = t^{-p\alpha}\int_{B_R}(C - k|x|^2 t^{-2\sigma})^{\frac{p}{m-1}}\,dx \\ &= t^{-p\alpha}\int_{B_R}(C - k|y|^2)^{\frac{p}{m-1}} t^{\sigma d}\,dy \\ &= C_{m,p,k} t^{-p\alpha+\sigma d} = C_{m,p,k} t^{\alpha(1-p)} = C_{m,p,k} t^{-\frac{d(p-1)}{d(m-1)+2}},\end{aligned}$$

where we considered the change of variables $y = x/t^{\sigma}$, with $dx = t^{\sigma d}\,dy$. Then, by comparison with (34), we conclude that our estimate is sharp.

4.3 Comparison with previous work

We now compare the results of our method with estimates in the literature. In [14], using phase-plane analysis, scaling techniques, and self-similarity, it was shown that

$$\|u\|_{L^p(\mathbb{R}^d)} \le C\|u_0\|_{L^q(\mathbb{R}^d)}^{\sigma(p,q)} t^{-\alpha(p,q)}$$

with

$$\alpha(p,q) = \frac{d(p-q)}{p(d(m-1)+2q)}, \quad \sigma(p,q) = \frac{q(d(m-1)+2p)}{p(d(m-1)+2q)}.$$

In our case, we fix $q = 1$ to get

$$\|u\|_{L^p(\mathbb{R}^d)} \le C\|u_0\|_{L^1(\mathbb{R}^d)}^{\sigma(p,1)} t^{-\alpha(p,1)} = C\|u_0\|_{L^1(\mathbb{R}^d)}^{\frac{d(m-1)+2p}{p(d(m-1)+2)}} t^{-\frac{d(p-1)}{p(d(m-1)+2)}}$$

which yields the same estimate as in (34). Hence, our technique provides a different method to establish the results in [14] without relying on symmetry arguments.

4.4 Periodic solutions of the porous media equation

We now the deduce a similar estimate for the porous media equation on $\mathbb{T}^d$.

Proposition 3. *Let u solve (33) on the torus with $u \in C^\infty(\mathbb{T}^d \times [0,\infty))$. Then, there exists $T > 0$ such that the following holds*

$$\|u\|_{L^p(\mathbb{T}^d)} \le C\|u_0\|_{L^1(\mathbb{T}^d)}^{\frac{d(m-1)+2p}{p(d(m-1)+2)}} t^{-\frac{d(p-1)}{p(d(m-1)+2)}} \tag{39}$$

for all $t \in [0,T)$. For $t > T$, $\|u\|_{L^p(\mathbb{T}^d)} \le C\|u_0\|_{L^1(\mathbb{T}^d)}^{\frac{d(m-1)+2p}{p(d(m-1)+2)}} T^{-\frac{d(p-1)}{p(d(m-1)+2)}}$.

Proof. Fix $\gamma = (m+p-1)/2$, thus $2\gamma > p$. From (36), we have that $\frac{d}{dt}\int_{\mathbb{R}^d} u^p\,dx = -C\int_{\mathbb{R}^d} |\nabla(u^\gamma)|^2\,dx$. Then,

$$\begin{aligned}
\left(\int_{\mathbb{T}^d} u^p\,dx\right)^{\frac{\gamma}{p}} &\le \|u_0\|_{L^1(\mathbb{T}^d)}^{\gamma(1-\lambda)} \left(\int_{\mathbb{T}^d} u^{2^*\gamma}\,dx\right)^{\frac{\lambda}{2^*}} \\
&\le \|u_0\|_{L^1(\mathbb{T}^d)}^{\gamma(1-\lambda)} \left(\|u_0\|_{L^1(\mathbb{T}^d)} + \int_{\mathbb{T}^d} |D(u^\gamma)|^2\,dx\right)^{\frac{\lambda}{2}} \\
&\le \|u_0\|_{L^1(\mathbb{T}^d)}^{\gamma(1-\lambda)} \left(\|u_0\|_{L^1(\mathbb{T}^d)} - C\frac{d}{dt}\int_{\mathbb{T}^d} u^p\,dx\right)^{\frac{\lambda}{2}},
\end{aligned}$$

where $\lambda = \frac{d(p-1)(m+p-1)}{p(2+d(m+p-2))}$. Then, fixing $z(t) = \int_{\mathbb{T}^d} u^p\,dx$, we get the following differential inequality $\dot z \le C_1\|u_0\|_{L^1(\mathbb{T}^d)} - C_2\|u_0\|_{L^1(\mathbb{T}^d)}^{\frac{2\gamma(\lambda-1)}{\lambda}} z^\beta$, where $\beta = \frac{2\gamma}{\lambda p}$. Hence, by Lemma 2, there exists $T > 0$ such that z satisfies

$$z(t) \le C\|u_0\|_{L^1(\mathbb{T}^d)}^{\frac{d(m-1)+2p}{p(d(m-1)+2)}} t^{\frac{d(1-p)}{d(m-1)+2}}$$

for $t \in [0,T)$, which yields (39). For $t > T$,

$$\|u\|_{L^p(\mathbb{T}^d)} \le C\|u_0\|_{L^1(\mathbb{T}^d)}^{\frac{d(m-1)+2p}{p(d(m-1)+2)}} T^{-\frac{d(p-1)}{p(d(m-1)+2)}}.$$

■

5 Differential Inequalities

In this appendix, we present some of the estimates regarding differential inequalities used here.

Lemma 1. *Let $z : (0, \infty) \to (0, \infty)$ be a differentiable function satisfying the differential inequality*

$$z'(t) \leq -Cz(t)^{\beta} \tag{40}$$

for some constant $C > 0$ and $\beta > 1$. Then, z satisfies

$$z(t) \leq C_{\beta} t^{\frac{1}{1-\beta}}$$

for all $t > 0$.

Proof. Let $z \equiv z(t)$ and $\dot{z} \equiv z'(t)$. Since $\beta - 1 > 0$, multiplying both sides of (40) by $-(\beta-1)z^{-\beta}$ leads to $-(\beta-1)z^{-\beta}\dot{z} \geq (\beta-1)C$. Next, we observe that the left-hand side in the prior equation is $\frac{d}{dt}(z(t)^{1-\beta})$. Hence, integrating in time, we get

$$z(t)^{1-\beta} \geq z(0)^{1-\beta}(1 + z(0)^{\beta-1}(\beta-1)Ct).$$

Therefore,

$$\begin{aligned} z(t) &\leq \frac{z(0)}{(1 + z(0)^{\beta-1}(\beta-1)Ct)^{\frac{1}{\beta-1}}} \leq \frac{1}{(z(0)^{1-\beta} + (\beta-1)Ct)^{\frac{1}{\beta-1}}} \\ &\leq \frac{1}{((\beta-1)Ct)^{\frac{1}{\beta-1}}}. \end{aligned}$$

Hence, since $0 < z(0) < \infty$, z satisfies $z(t) \leq C_{\beta} t^{\frac{1}{1-\beta}}$ for some constant $C_{\beta} > 0$ depending on β. ■

Lemma 2. *Let $z : (0, \infty) \to (0, \infty)$ be a differentiable function satisfying the differential inequality*

$$\dot{z} \leq C_1 z^{\theta} - C_2 z^{\beta} \tag{41}$$

for constants $C_1, C_2 > 0$, and $1 \leq \theta < \beta$. Then, there exists $T > 0$ such that

$$z(t) \leq C_{\beta} t^{\frac{1}{1-\beta}}$$

for $t \in (0, T)$. Moreover, for $t > T$, $z(t) \leq C_{\beta} T^{\frac{1}{1-\beta}}$.

Proof. The function

$$z \mapsto C_1 z^\theta - C_2 z^\beta$$

has a single positive zero $\overline{z}$. Fix $z_0 > \overline{z}$ such that

$$C_1(z)^\theta - C_2(z)^\beta < -\tilde{C} z^\beta$$

for $z > \tilde{z}$. Consider the solution $z_*(t)$ of

$$\dot{z}_* = -\tilde{C} z_*^\beta$$

defined on $(0, +\infty)$ with $\lim_{t\to 0} z_*(t) = +\infty$. Define T by

$$z_*(T) = \tilde{z}.$$

Then, if z satisfies (41), we have $z(t) \leq z_*(t)$ for $t \leq T$ and $z_*(t) \leq \tilde{z}$ for $t \geq T$. Thus, by computing z_* and then $\tilde{z}$ as a function of T, we conclude that $z(t) \leq C t^{\frac{1}{1-\beta}}$ for all $t \in (0, T)$ and $z(t) \leq C T^{\frac{1}{1-\beta}}$ for $t > T$. ■

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